Outline

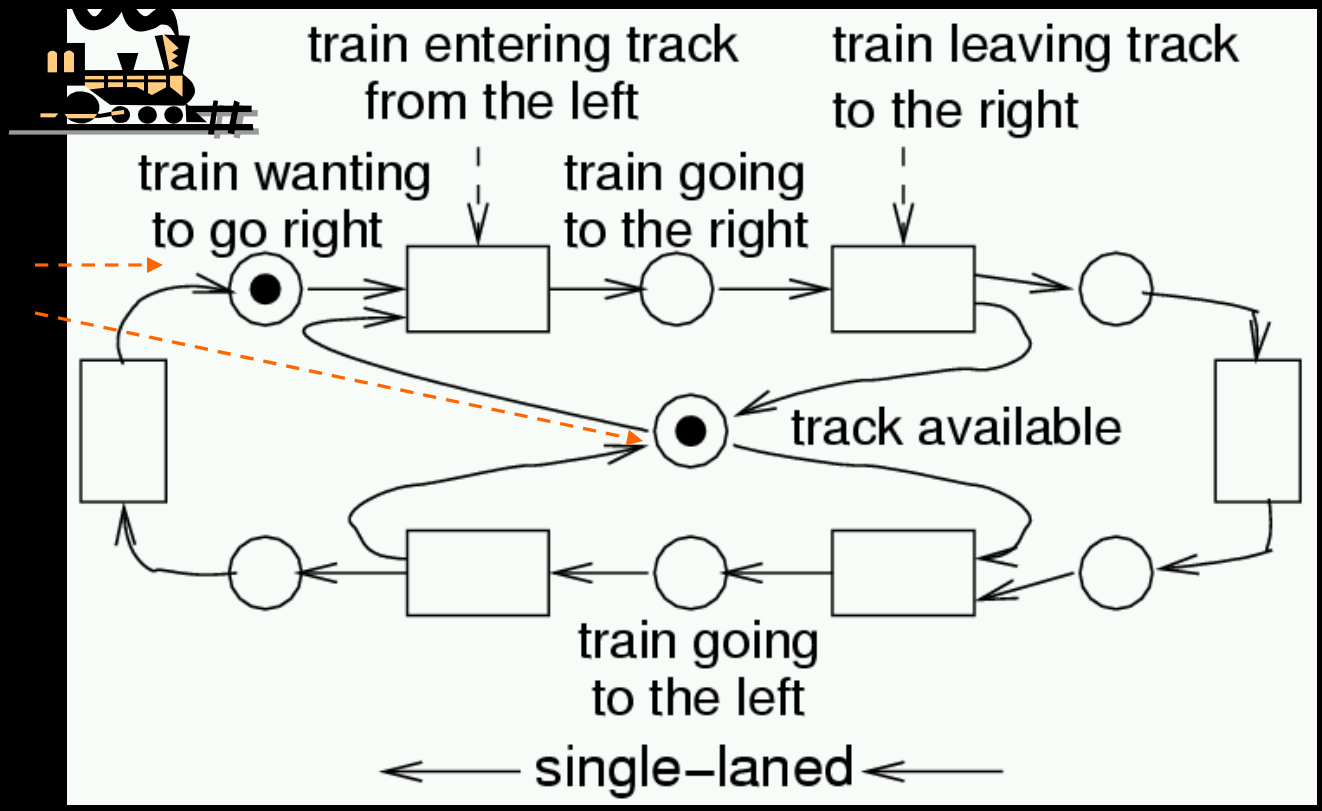
- **Petri nets**
  - **Introduction**
  - **Examples**
  - **Properties**
  - **Analysis techniques**

Petri Nets (PNs)

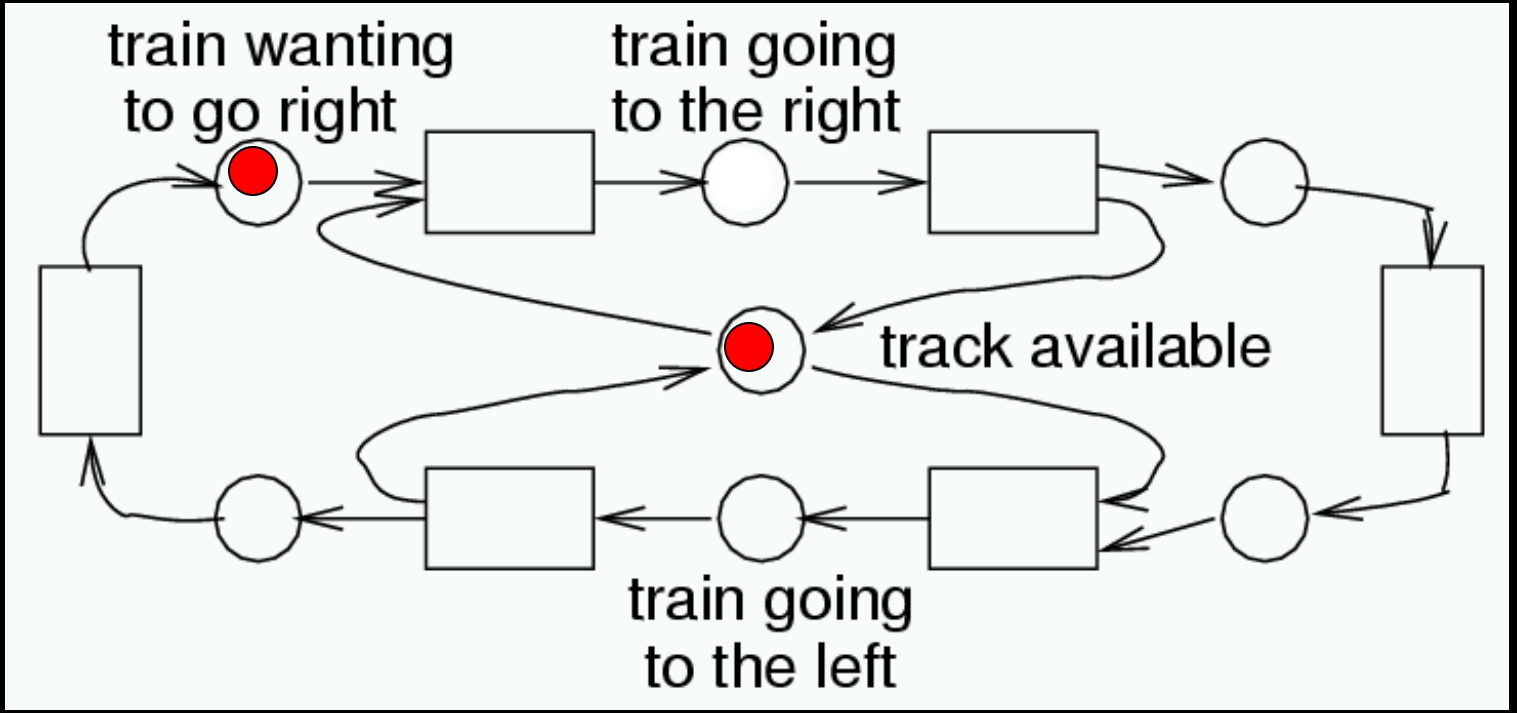
- **Model introduced by C.A. Petri in 1962**
  - **Ph.D. Thesis: "Communication with Automata"**
- **Applications: distributed computing, manufacturing, control, communication networks, transportation…**
- **PNs describe explicitly and graphically:**
  - **sequencing/causality**
  - **conflict/non-deterministic choice**
  - **concurrency**
- **Basic PN model**
  - **Asynchronous model (partial ordering)**
  - **Main drawback: no hierarchy**

Example: Synchronization at single track rail segment

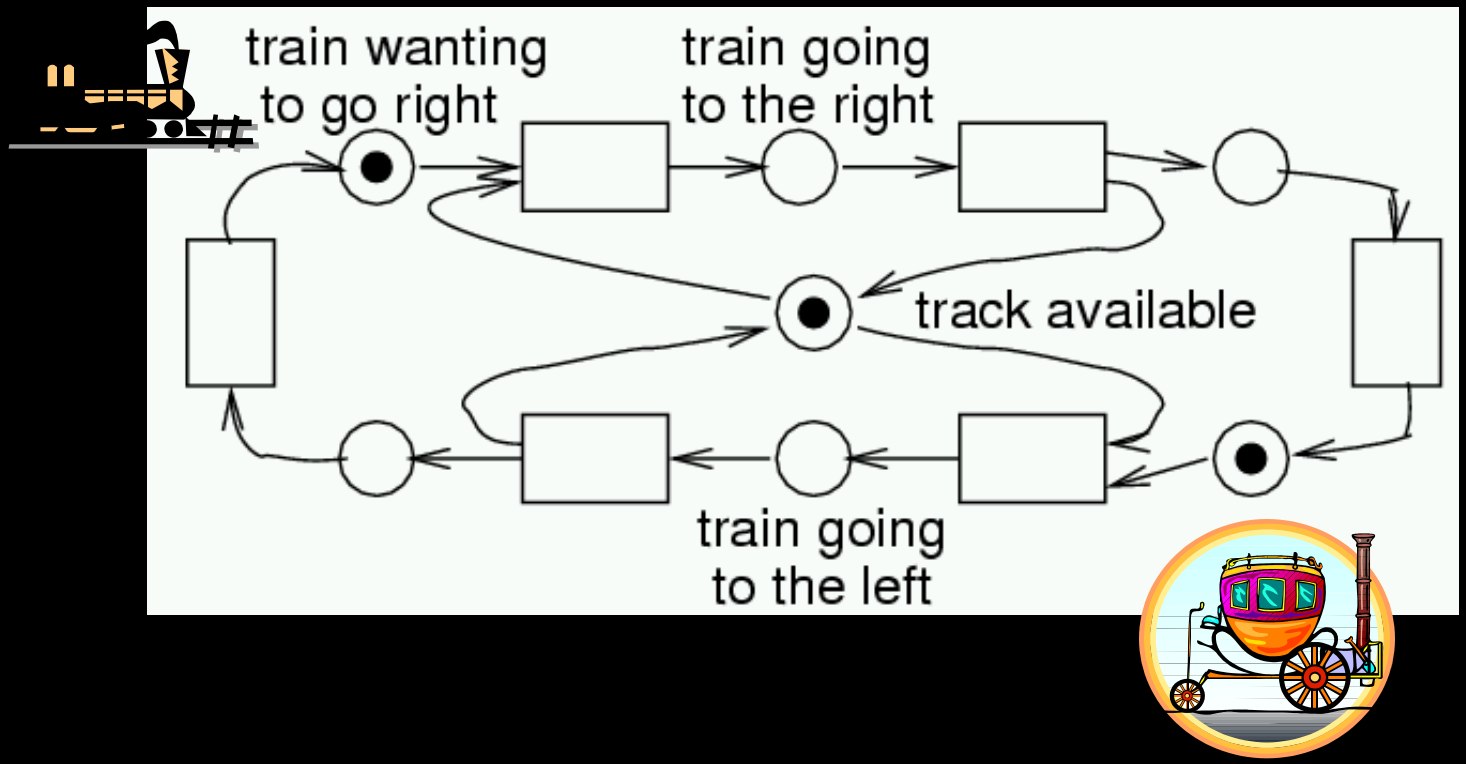
- „Preconditions“



Playing the „token game“

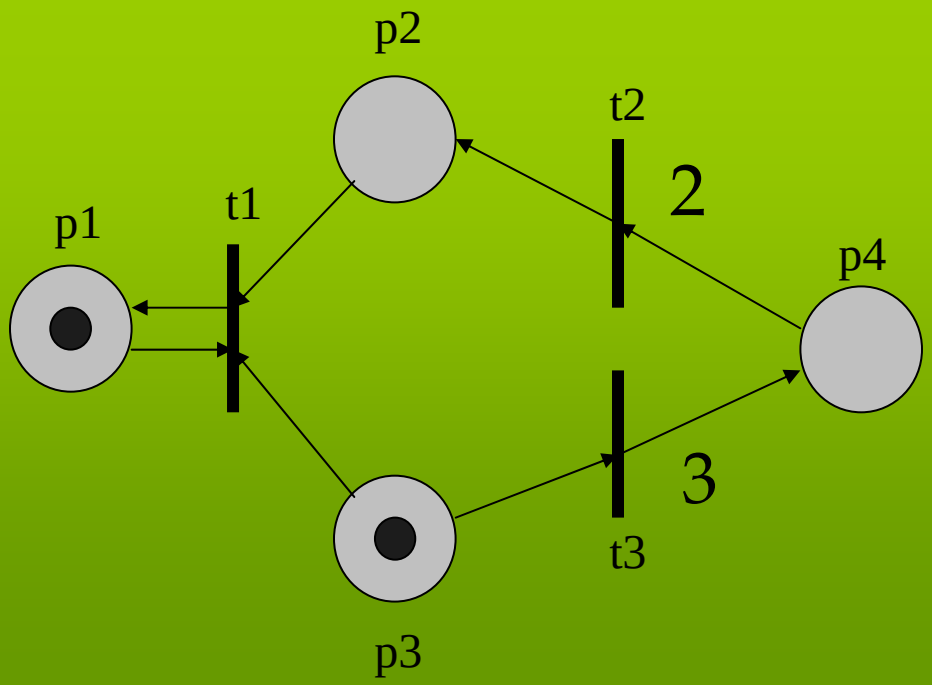


Conflict for resource „track“



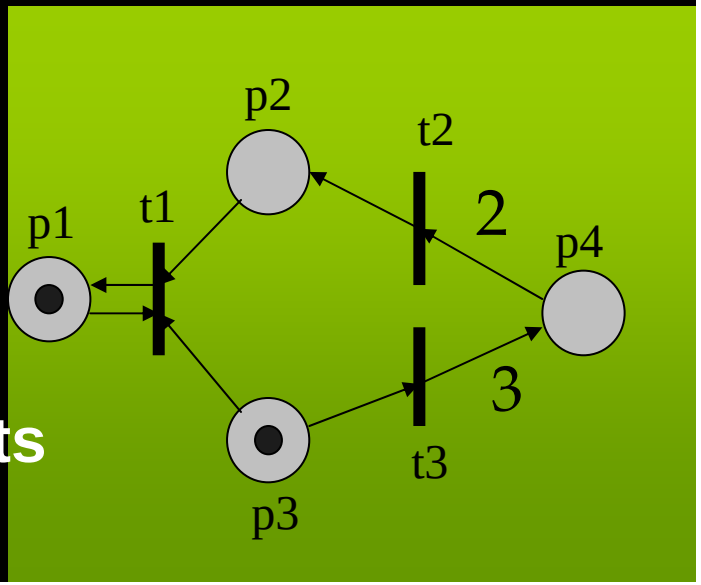
Petri Net Graph

- **Bipartite weighted directed graph:**
  - **Places: circles**
  - **Transitions: bars or boxes**
  - **Arcs: arrows labeled with weights**
- **Tokens: black dots**



Petri Net

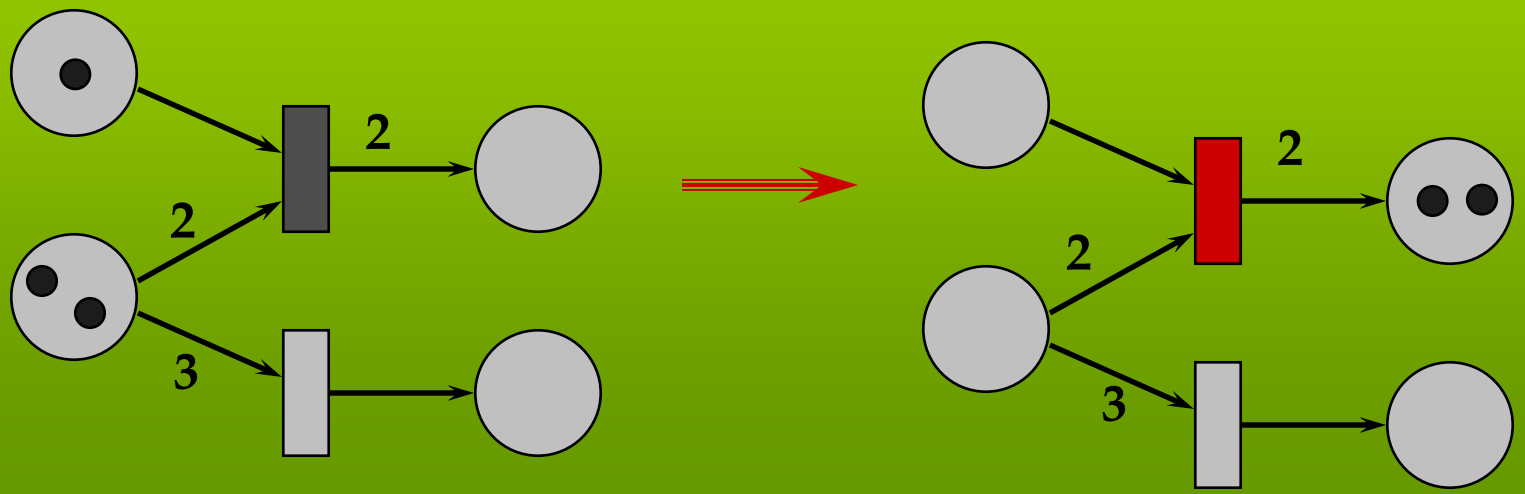
- **A PN (N,M_0) is a Petri Net Graph N**
  - **places: represent distributed state by holding tokens**
    - **marking (state) M is an n-vector (m_1,m_2,m_3…), where m_i is the non-negative number of tokens in place p_i.**
    - **initial marking (M_0) is initial state**
  - **transitions: represent actions/events**
    - **enabled transition: enough tokens in predecessors**
    - **firing transition: modifies marking**
- **…and an initial marking M_0.**



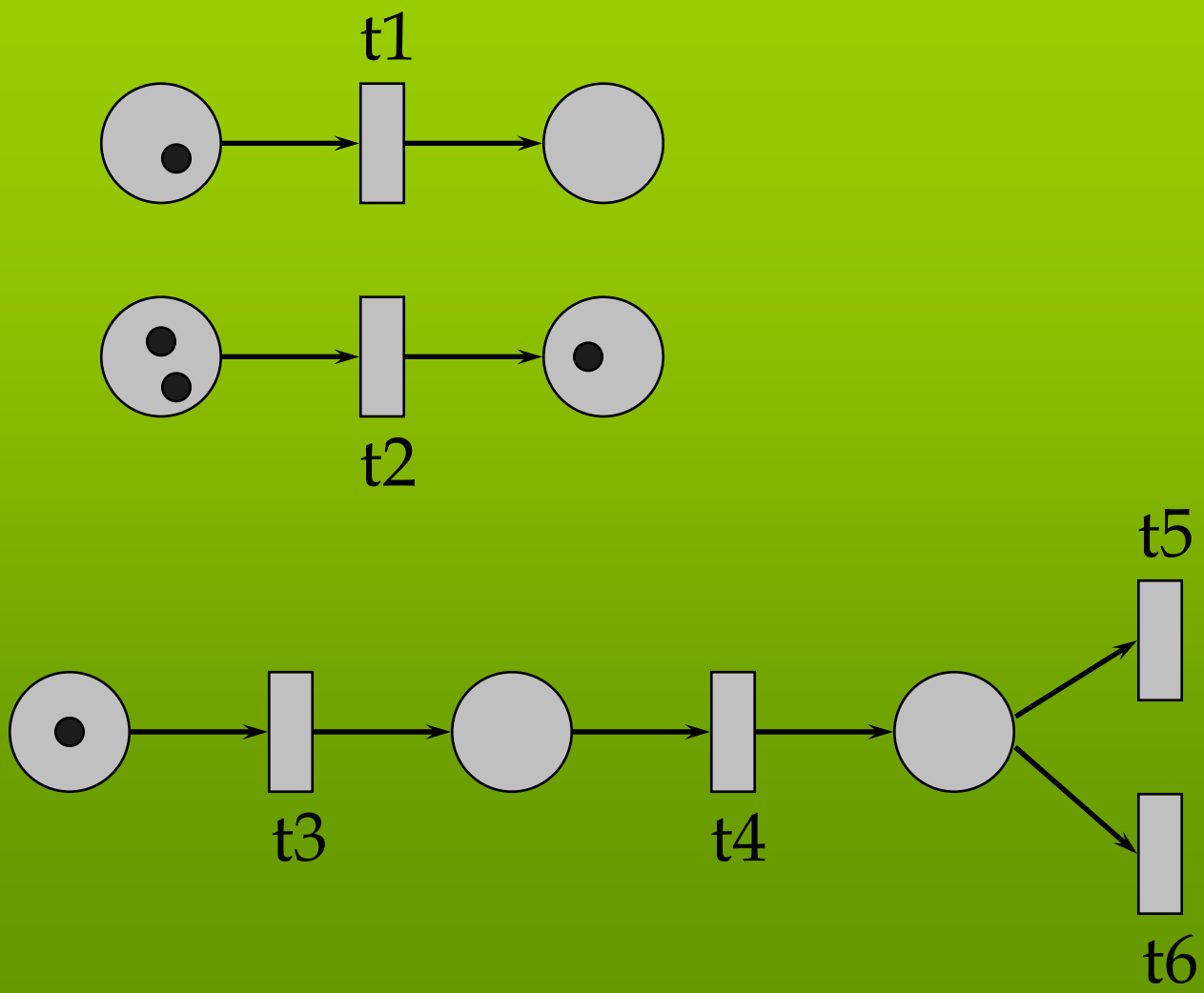
Places/Transitions: conditions/events

Transition firing rule

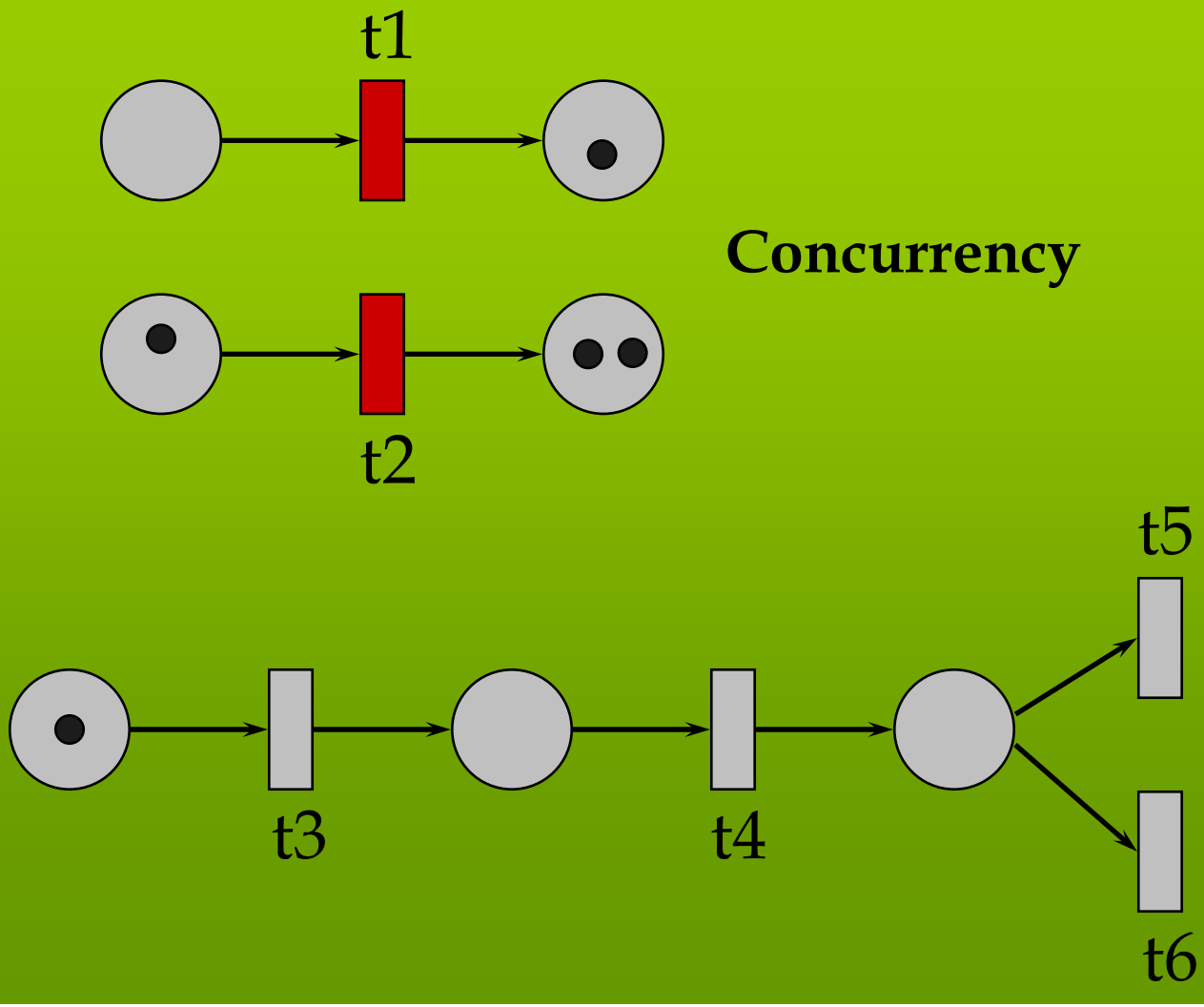
- **A marking is changed according to the following rules:**
  - **A transition is enabled if there are enough tokens in each input place**
  - **An enabled transition may or may not fire**
  - **The firing of a transition modifies marking by consuming tokens from the input places and producing tokens in the output places**



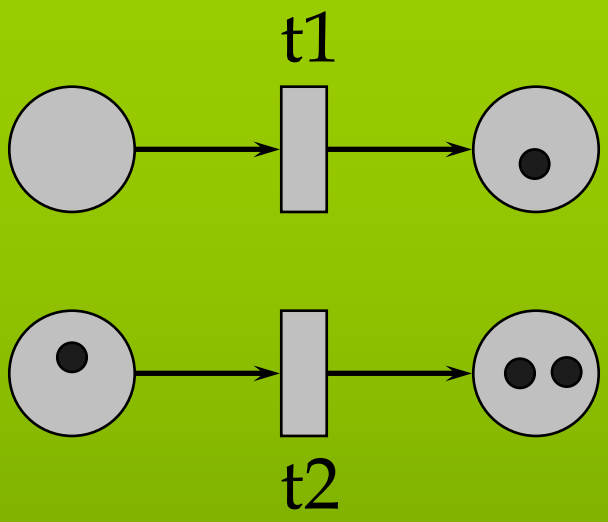
Concurrency, causality, choice



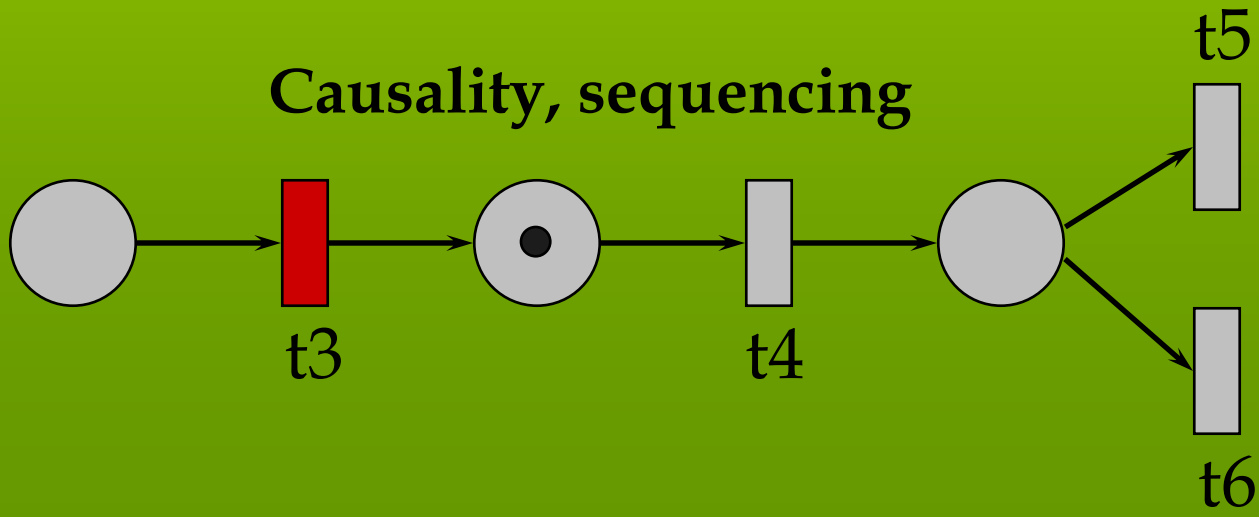
Concurrency, causality, choice



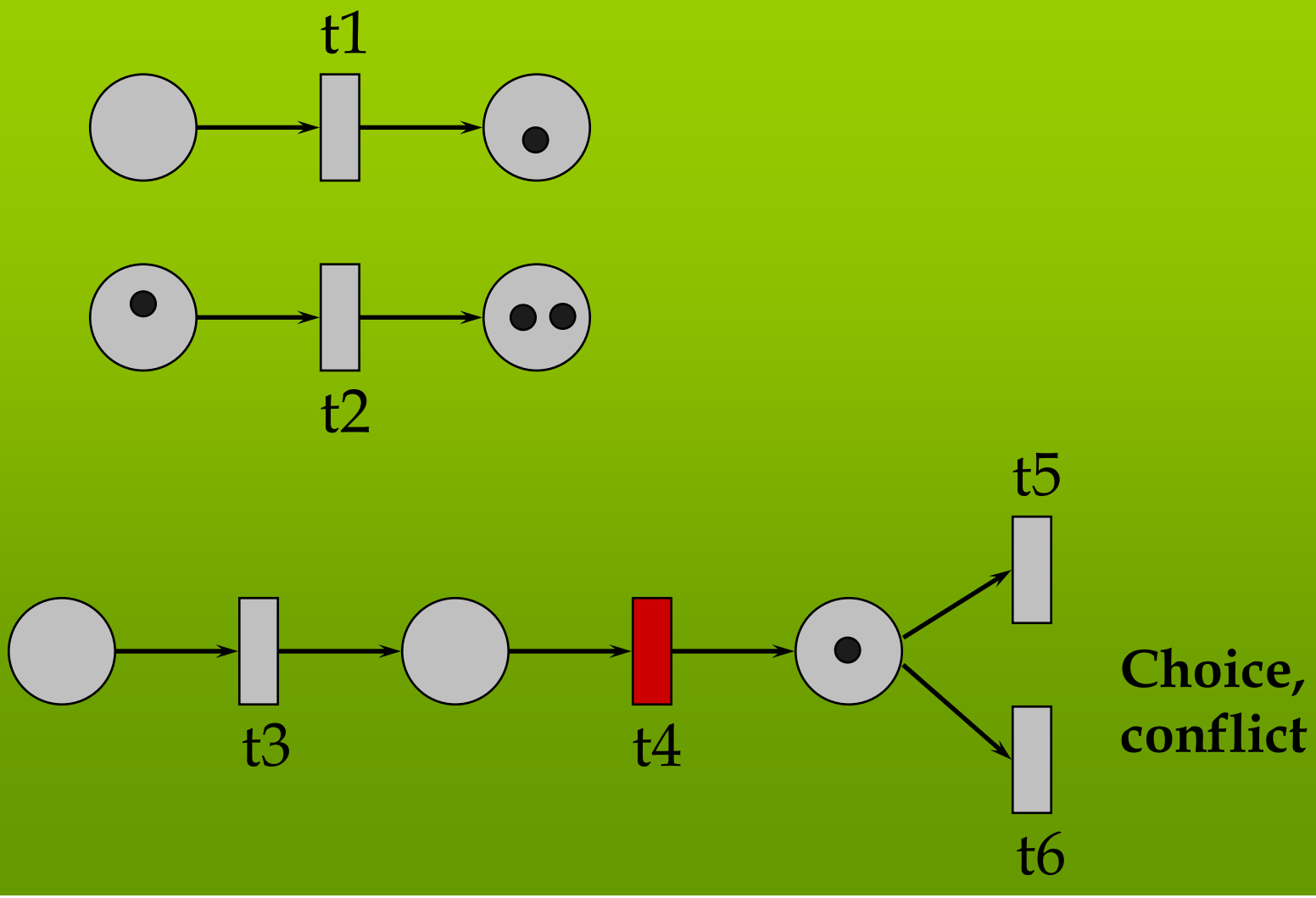
Concurrency, causality, choice



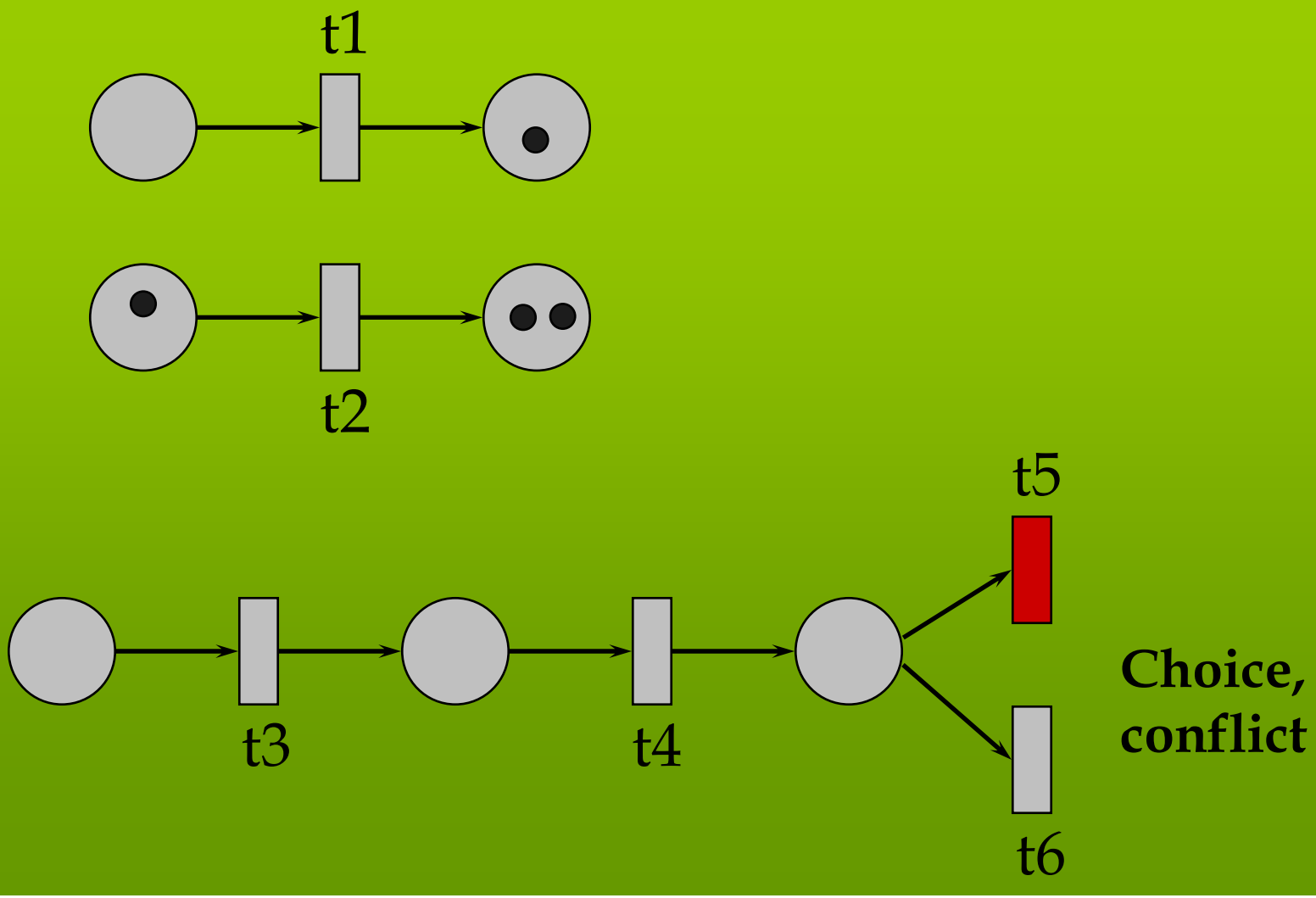
Causality, sequencing



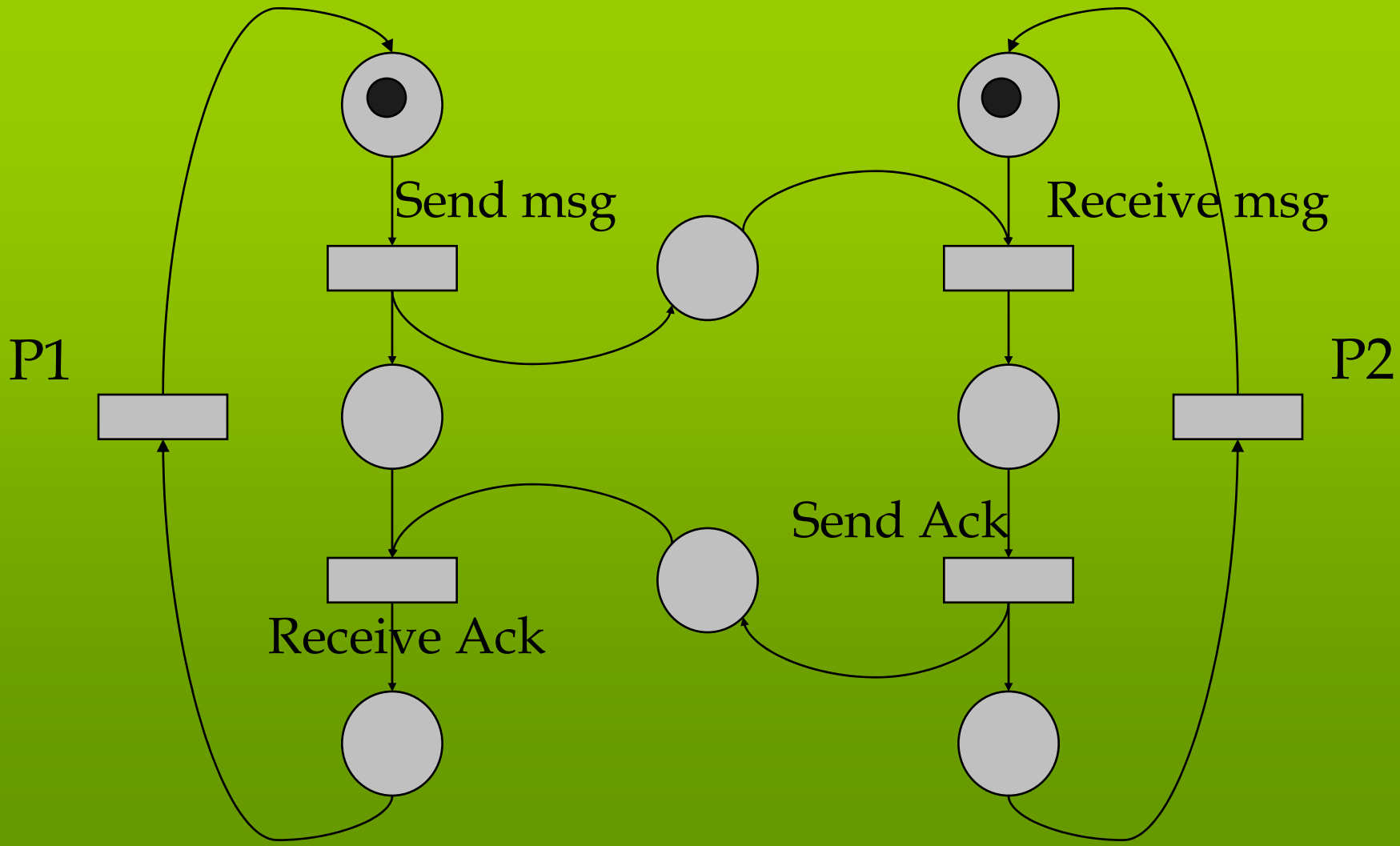
Concurrency, causality, choice



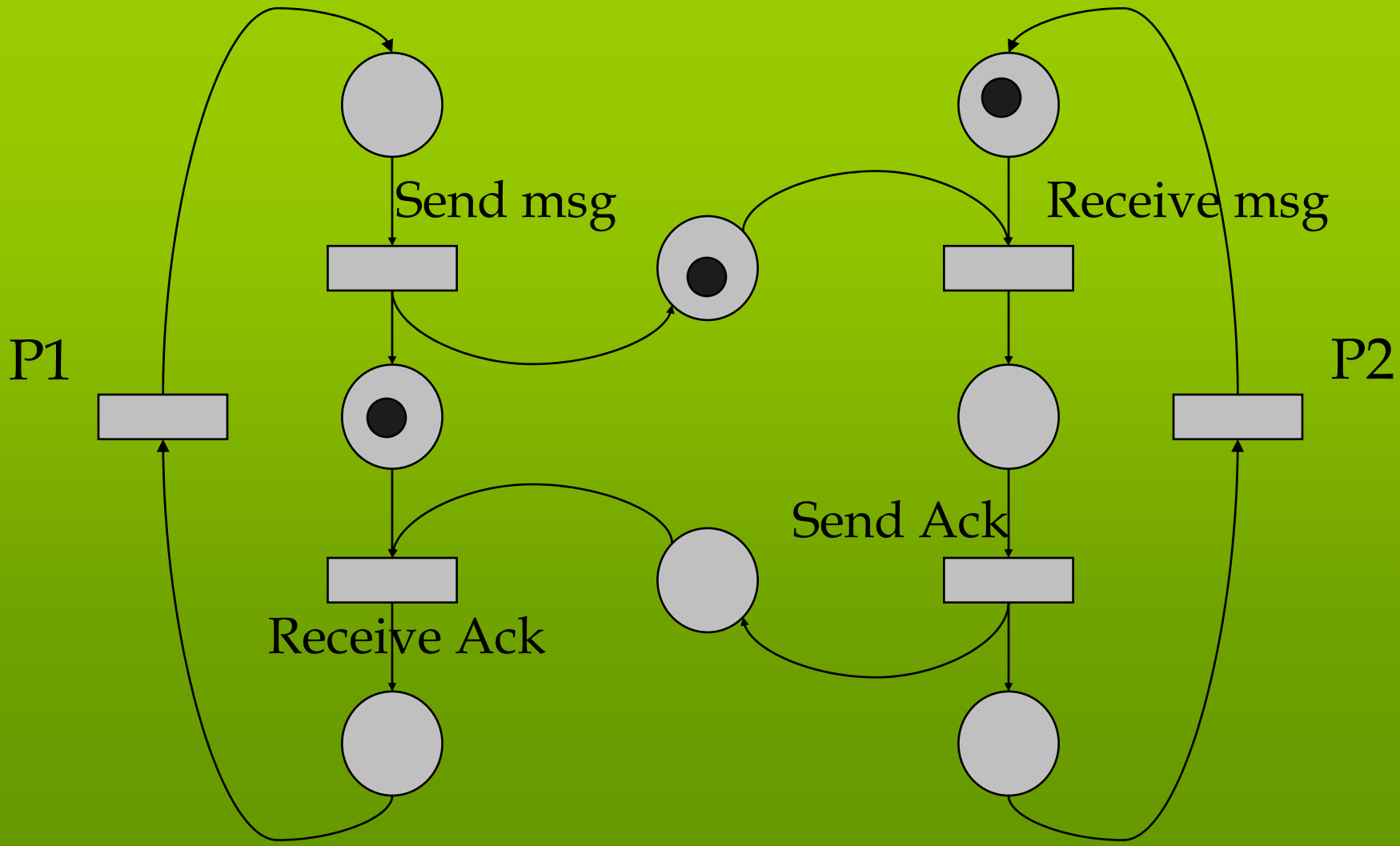
Concurrency, causality, choice



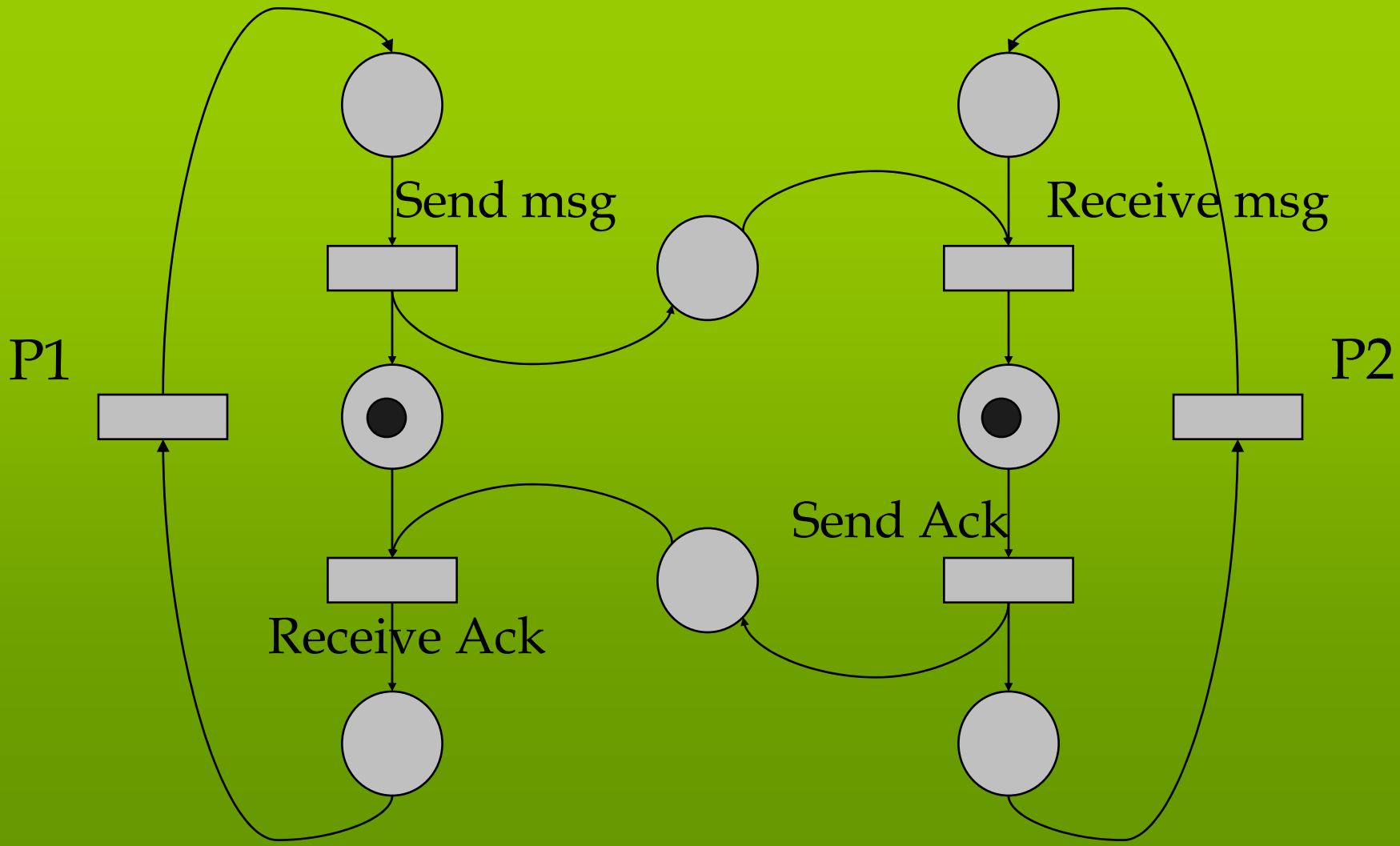
Communication Protocol



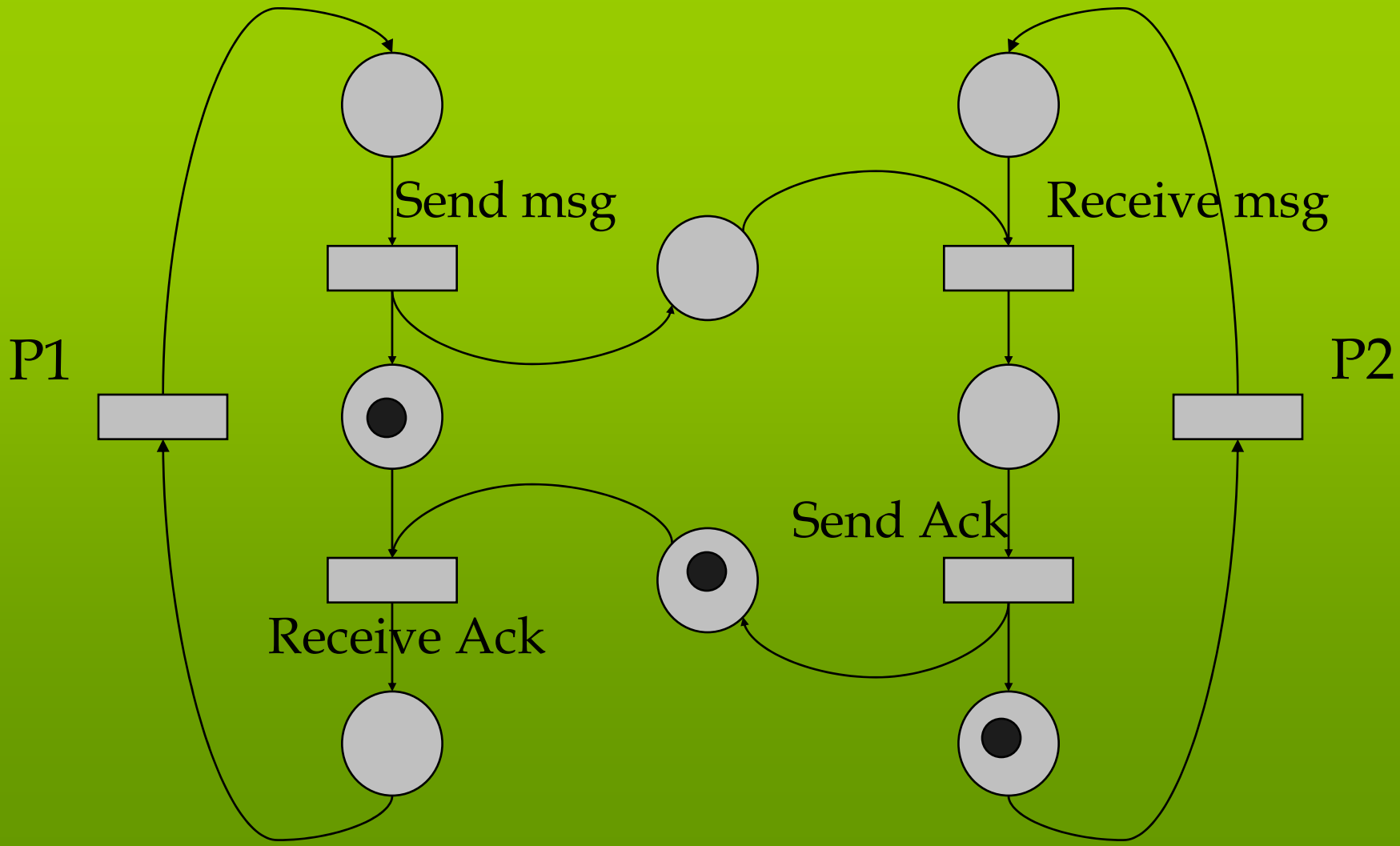
Communication Protocol



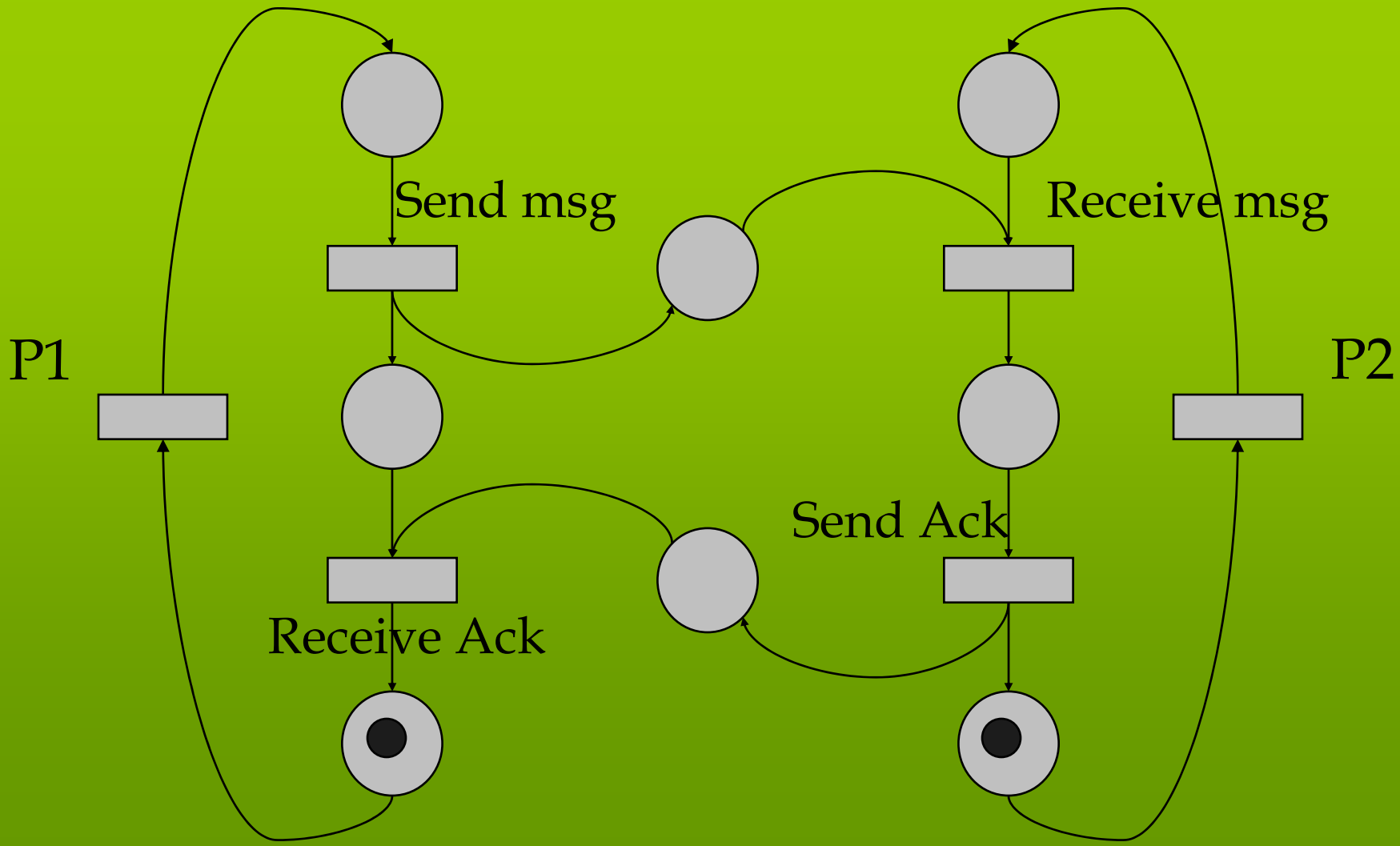
Communication Protocol



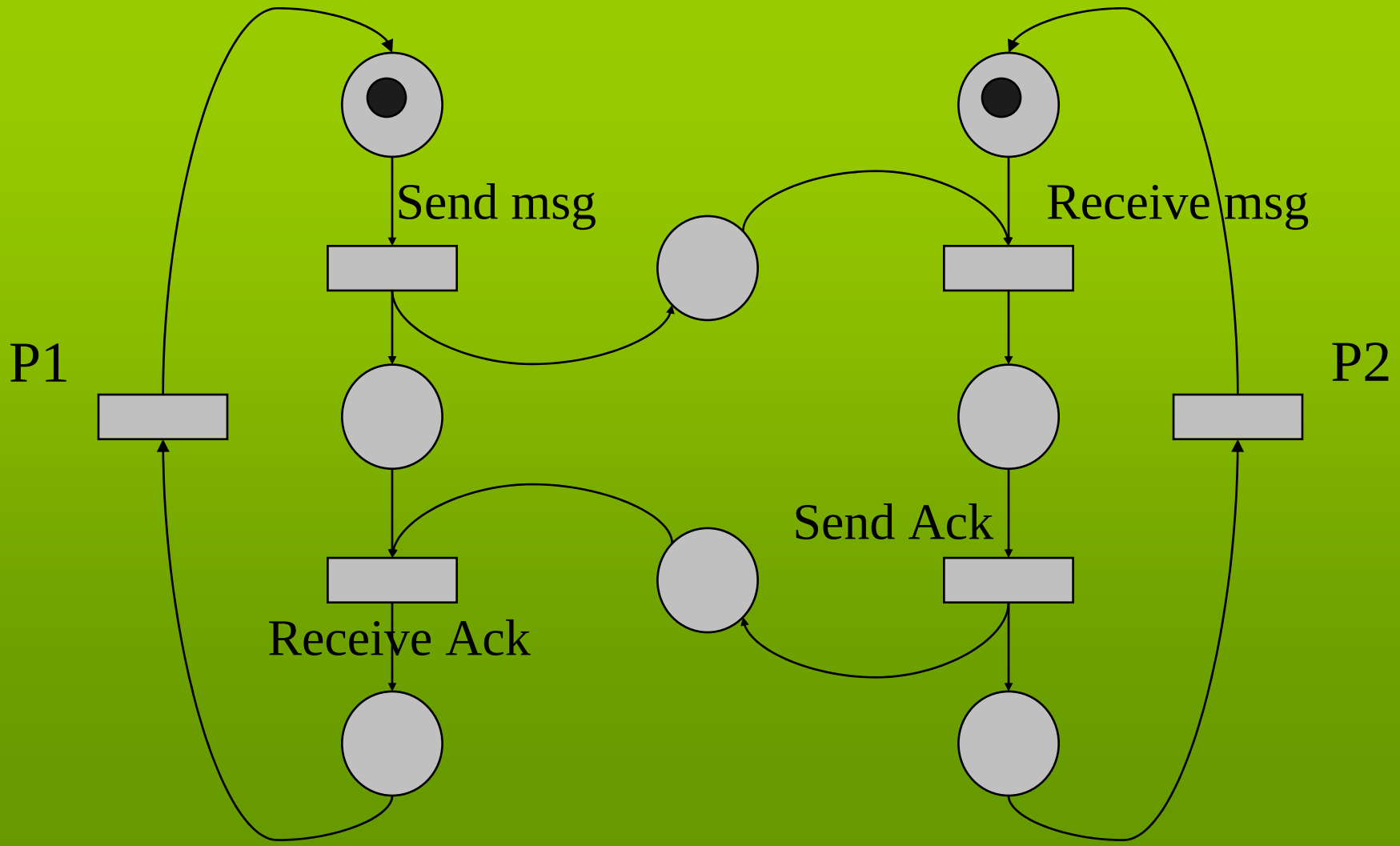
Communication Protocol



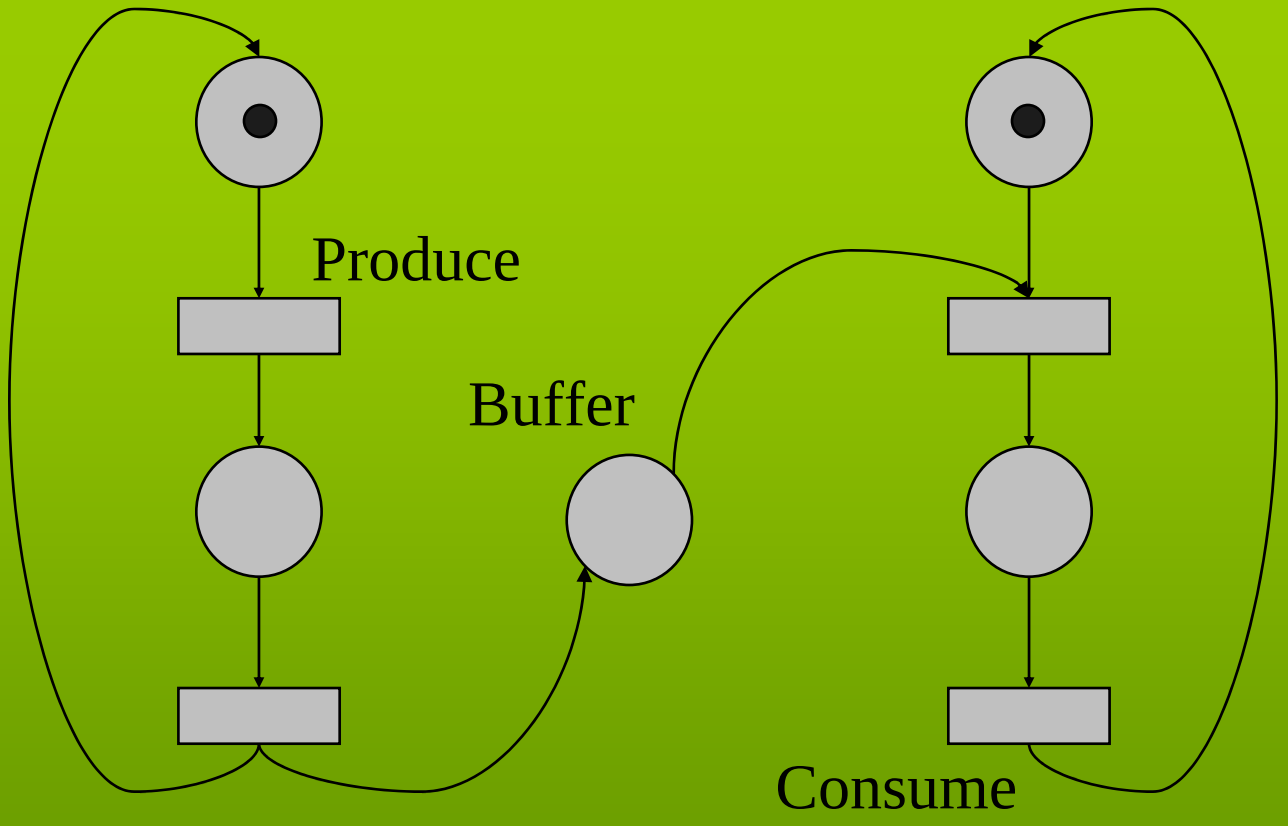
Communication Protocol



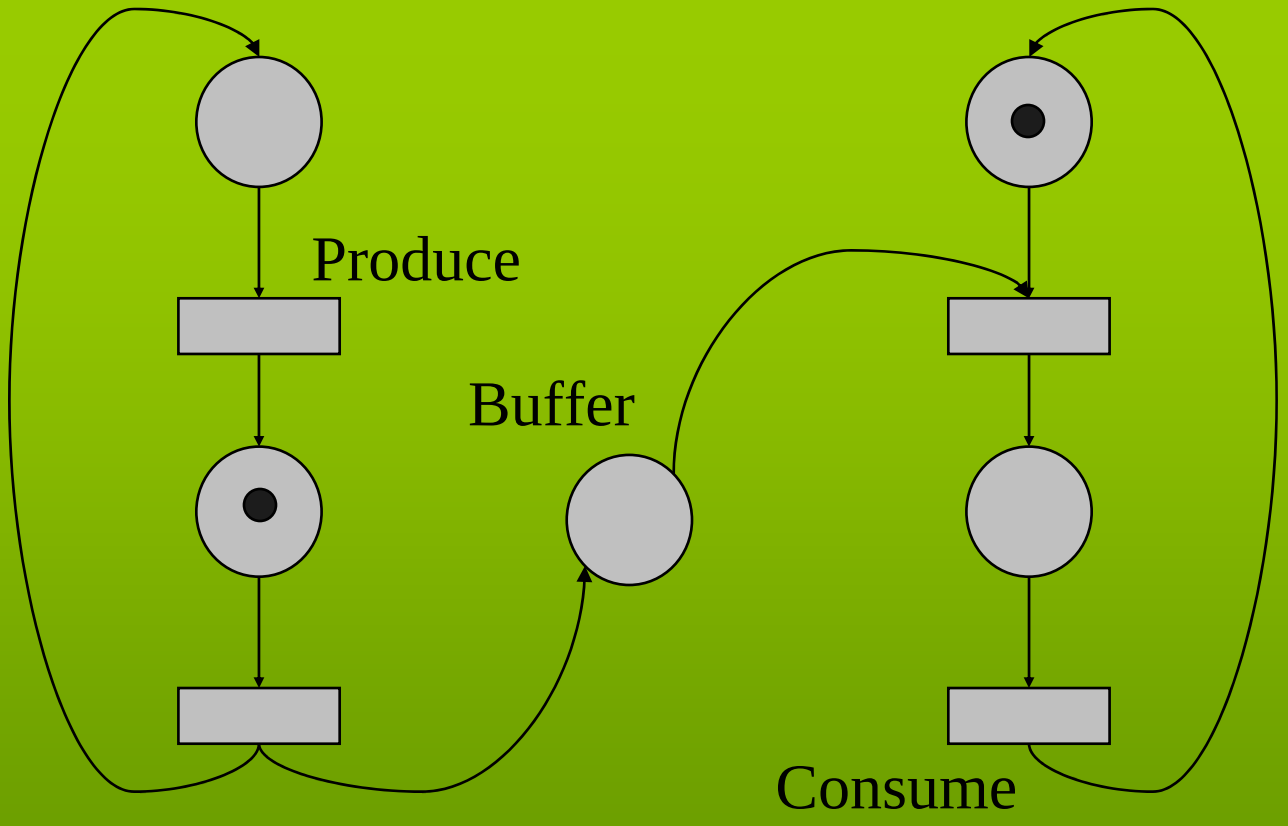
Communication Protocol



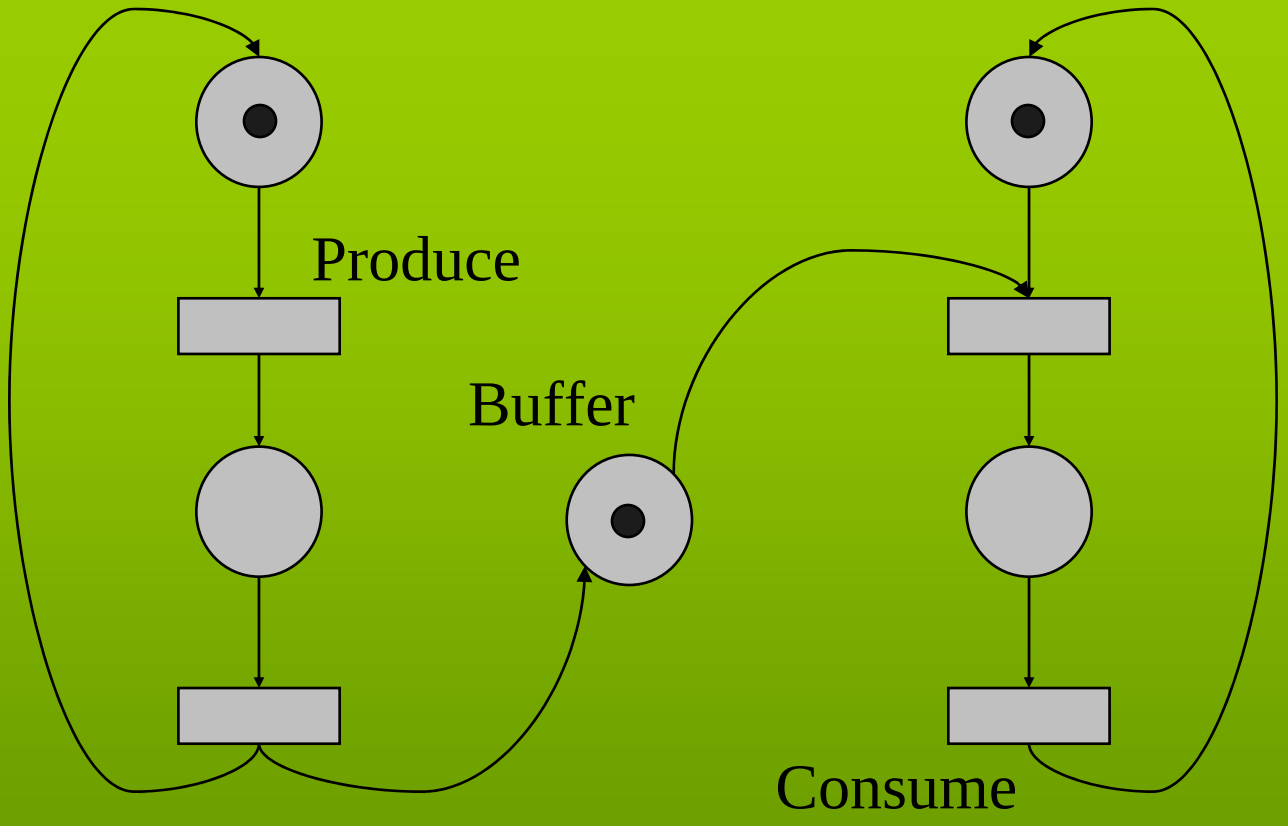
Producer-Consumer Problem



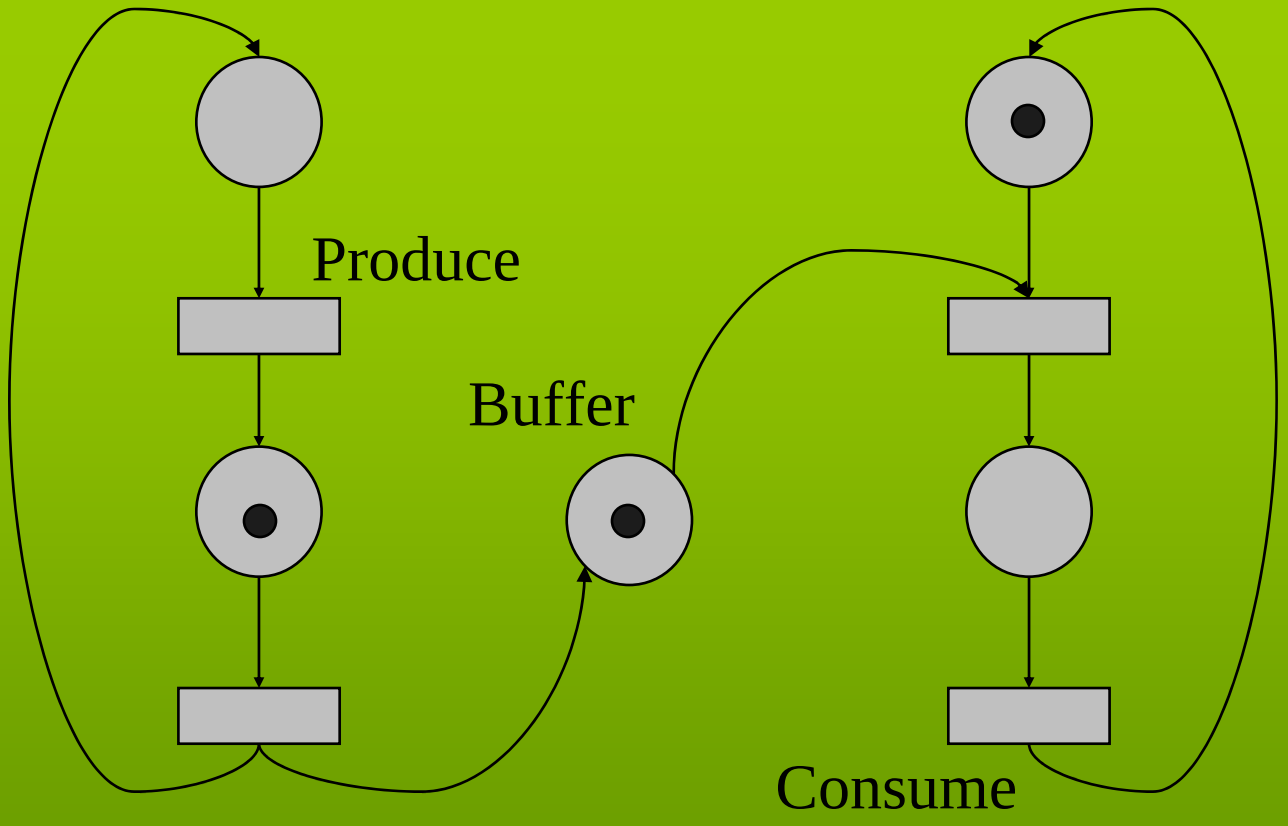
Producer-Consumer Problem



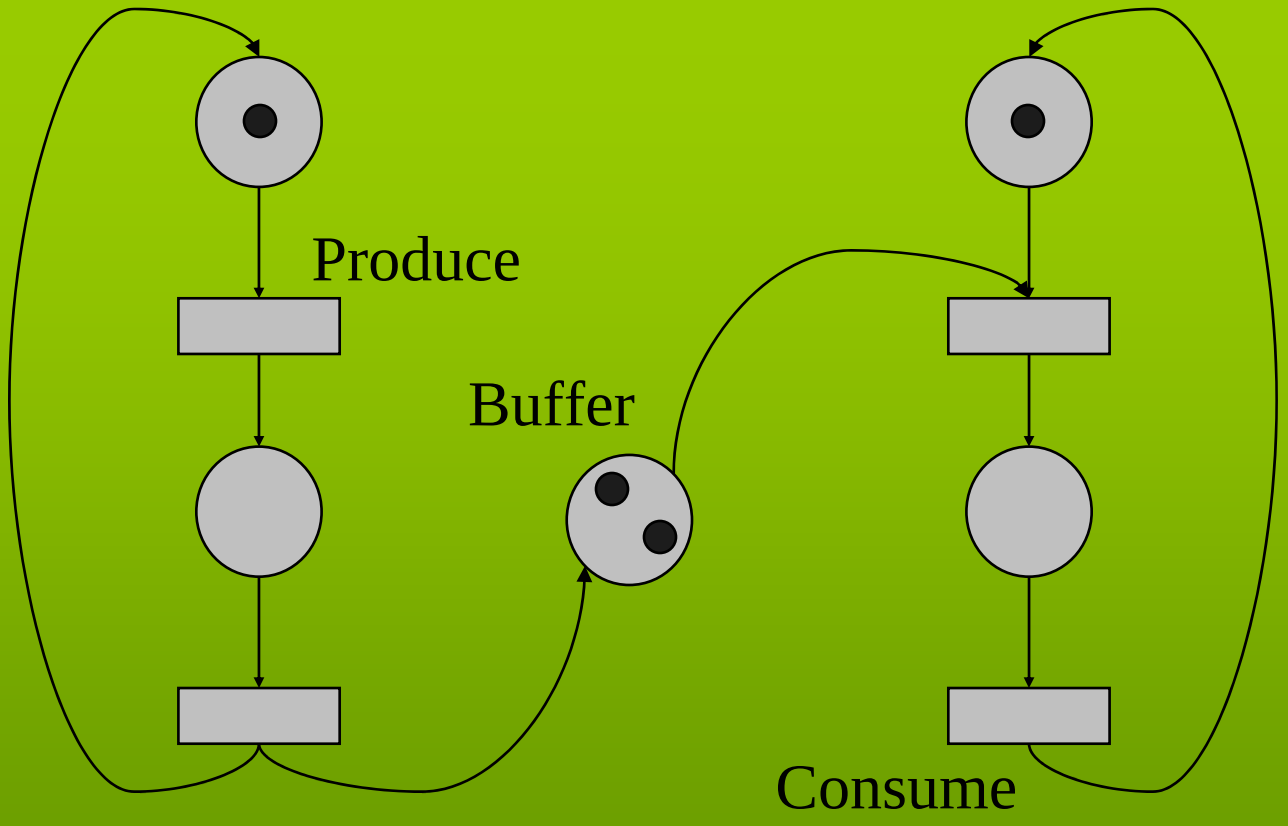
Producer-Consumer Problem



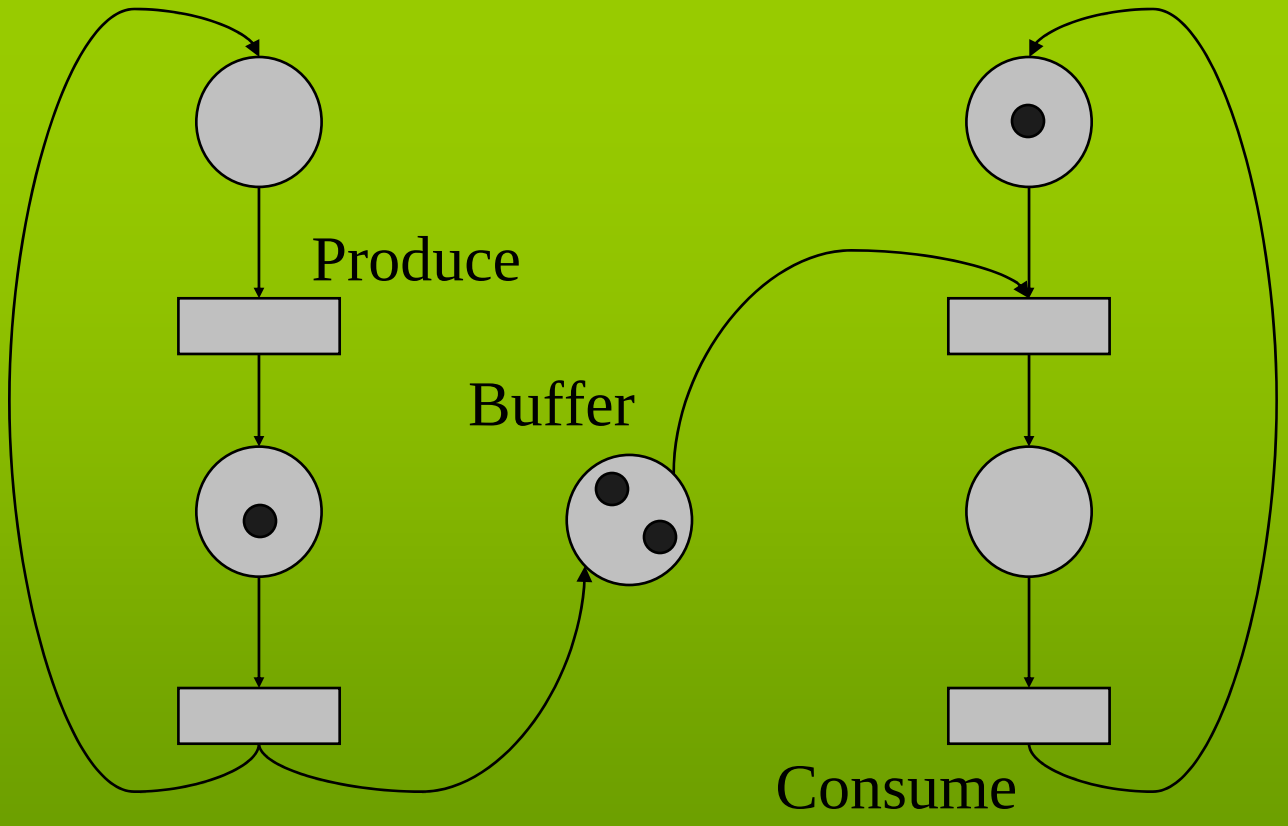
Producer-Consumer Problem



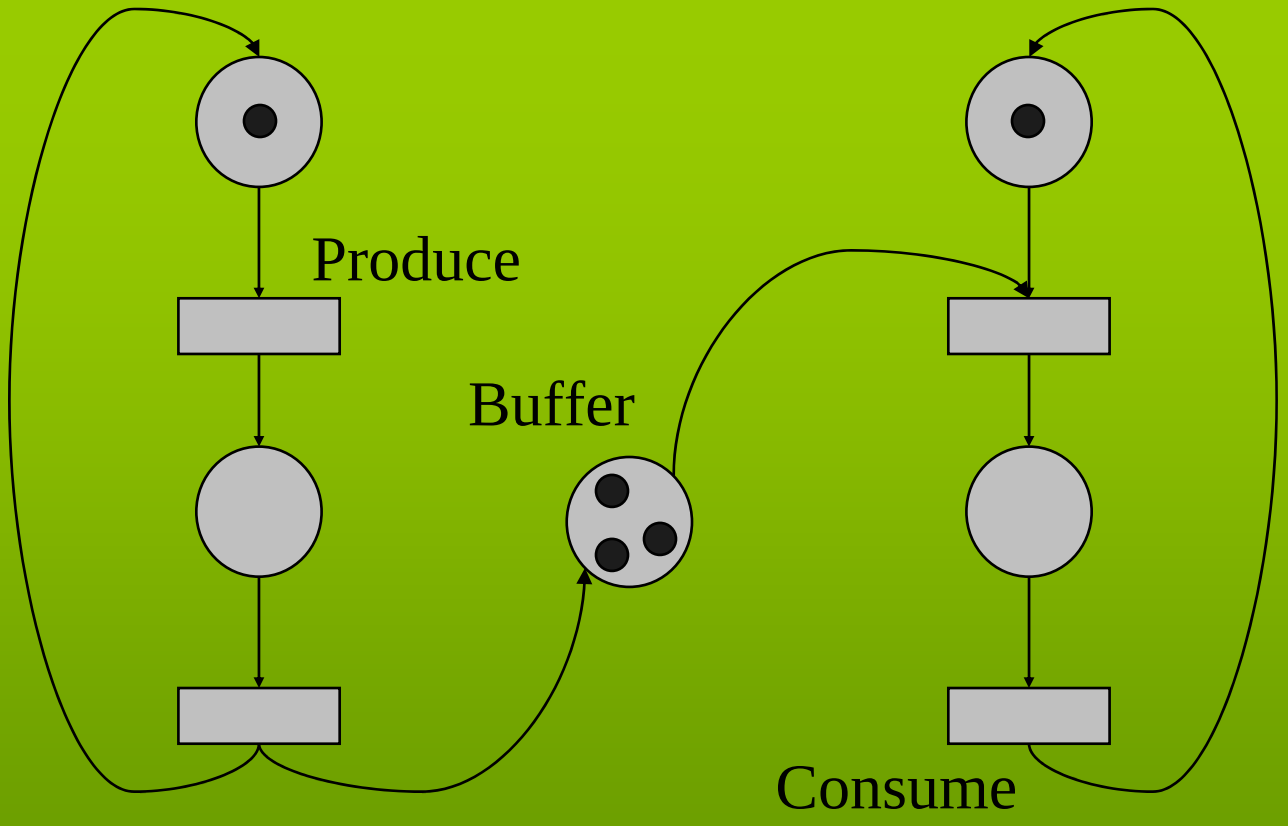
Producer-Consumer Problem



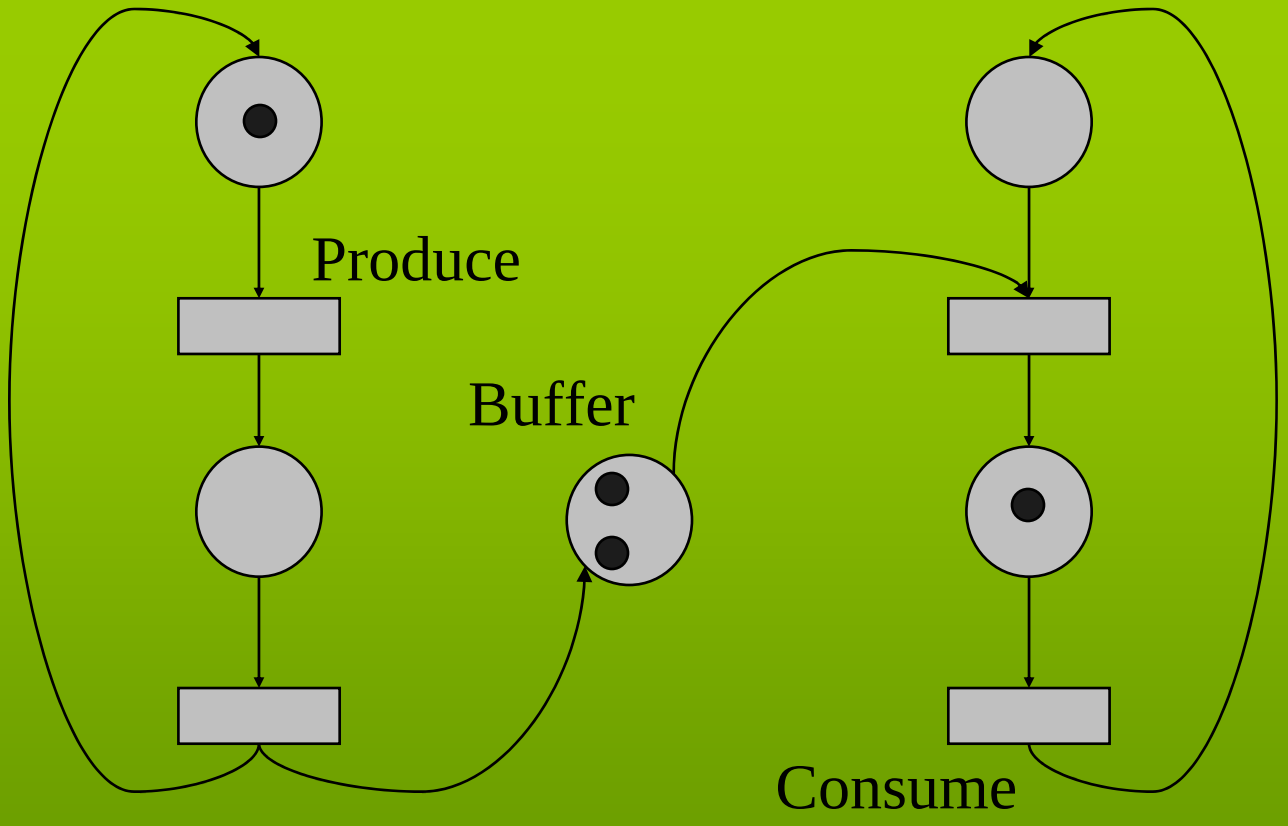
Producer-Consumer Problem



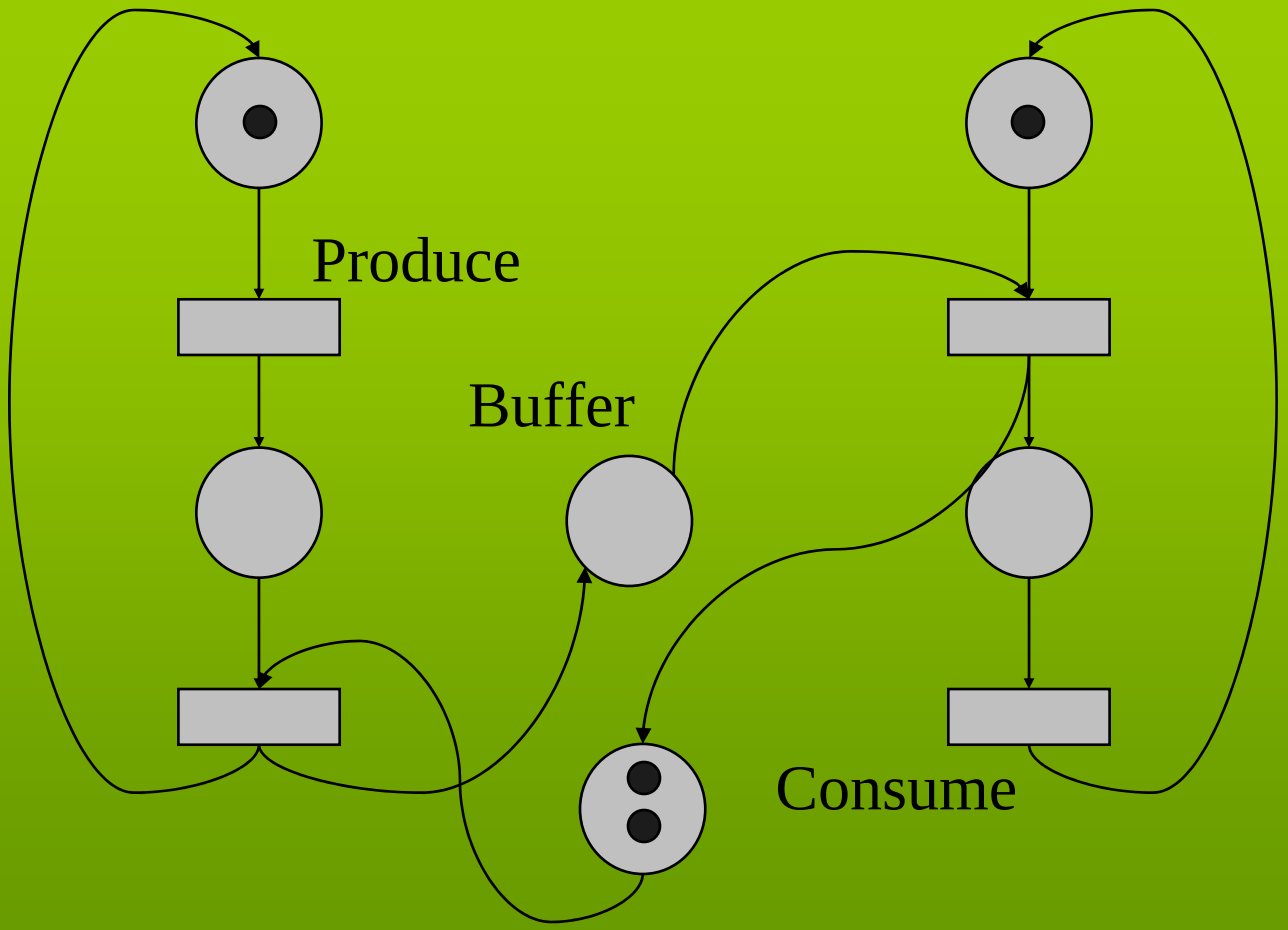
Producer-Consumer Problem



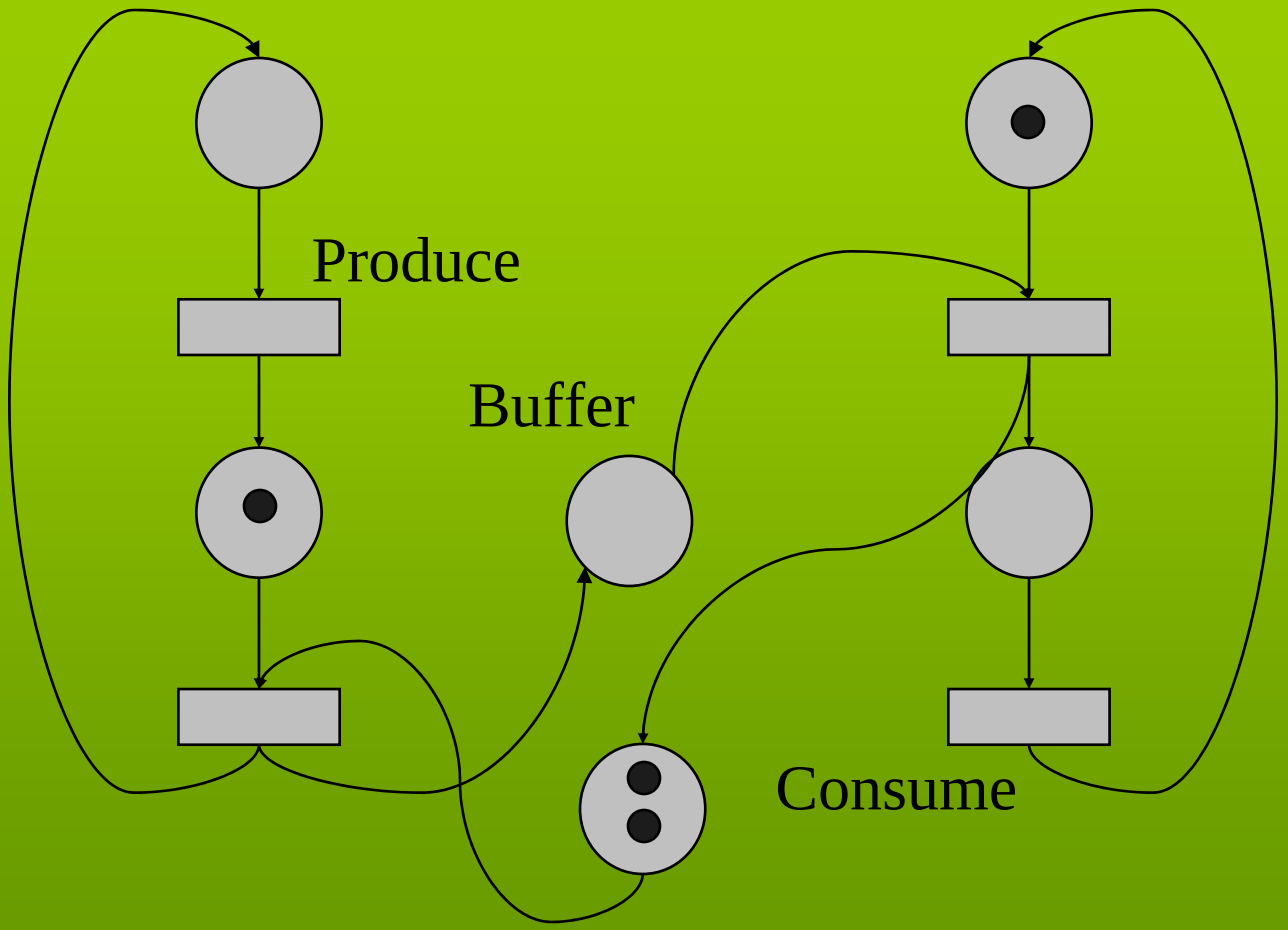
Producer-Consumer Problem



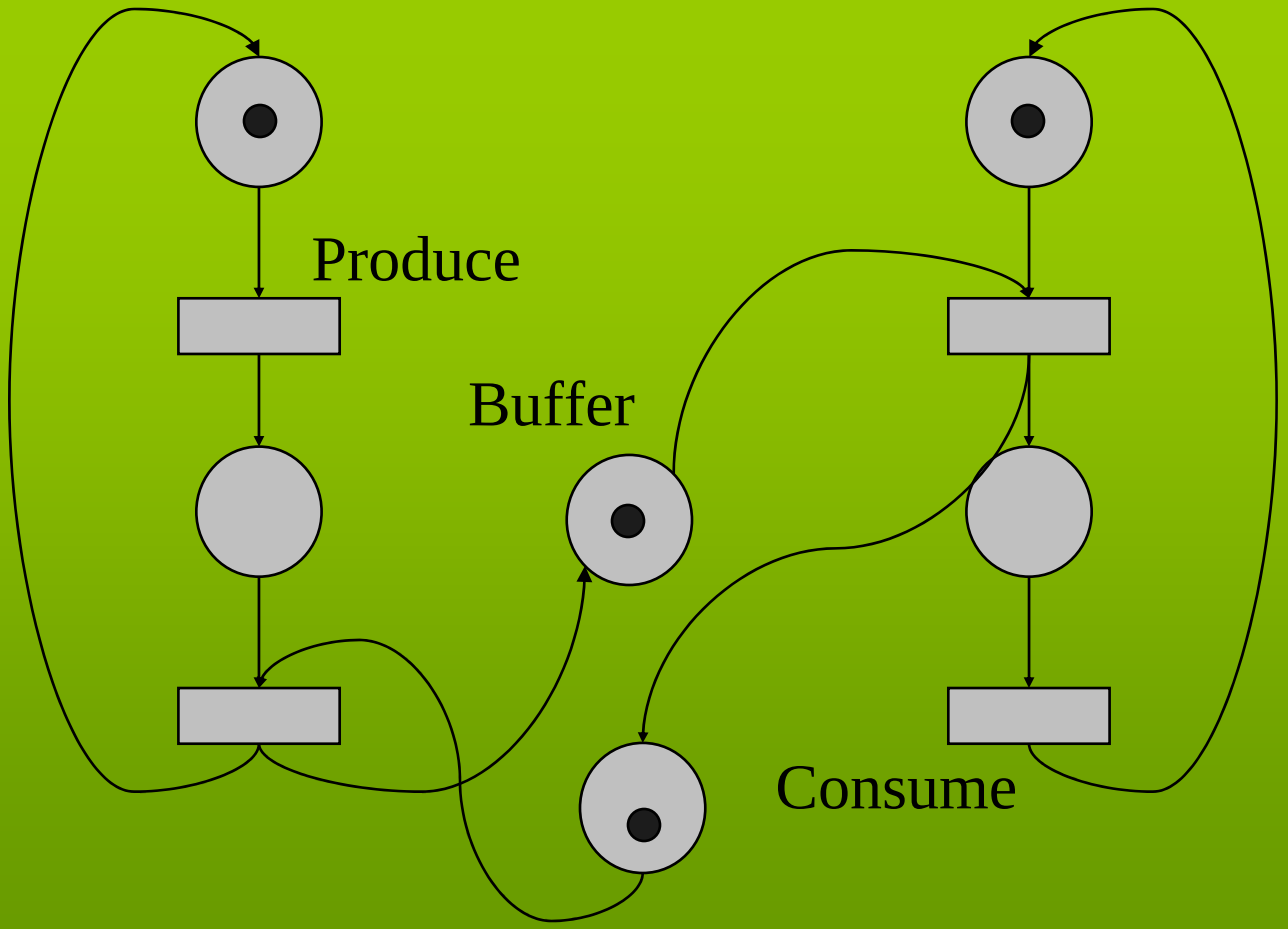
Producer-Consumer Problem



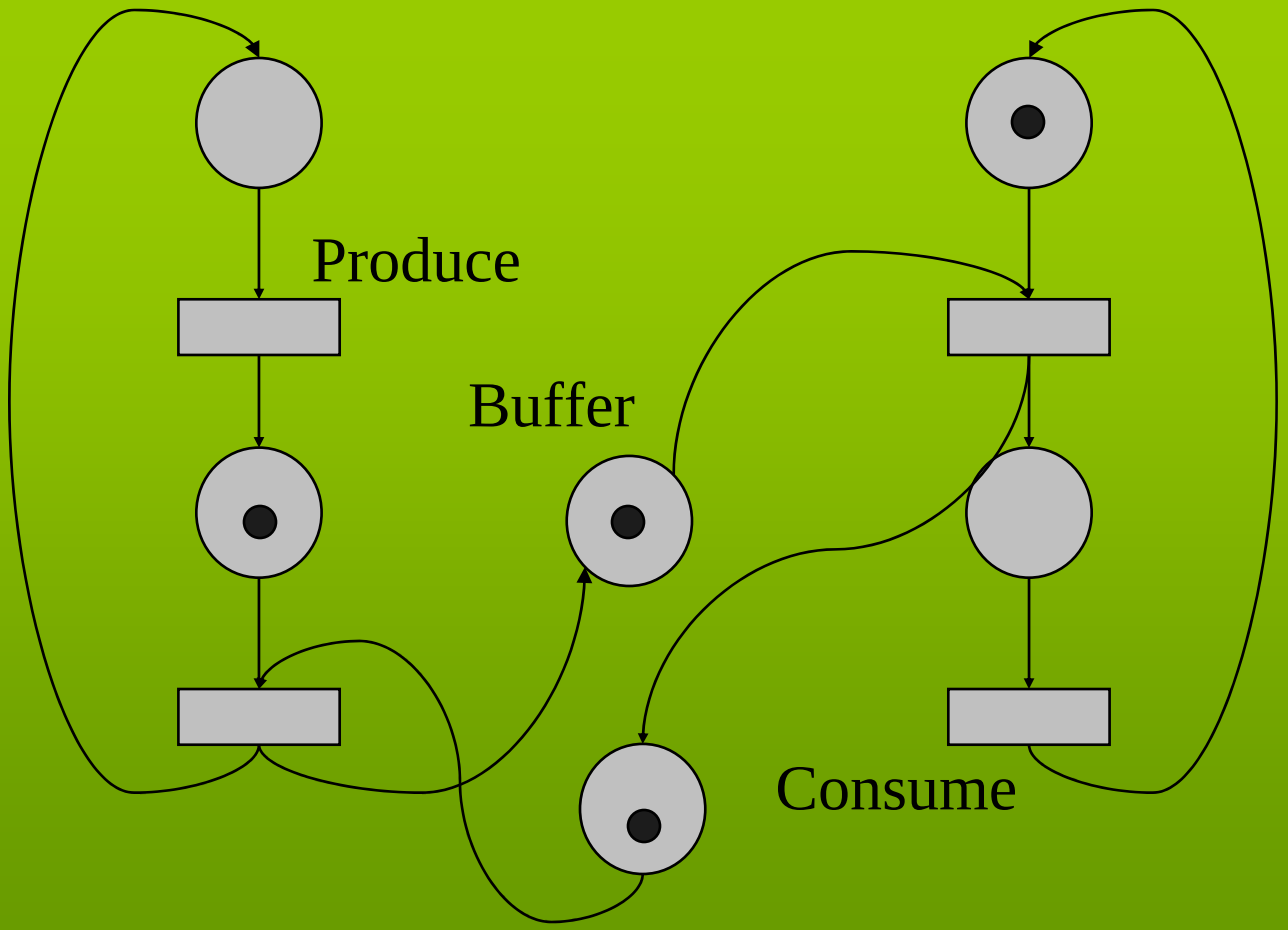
Producer-Consumer Problem



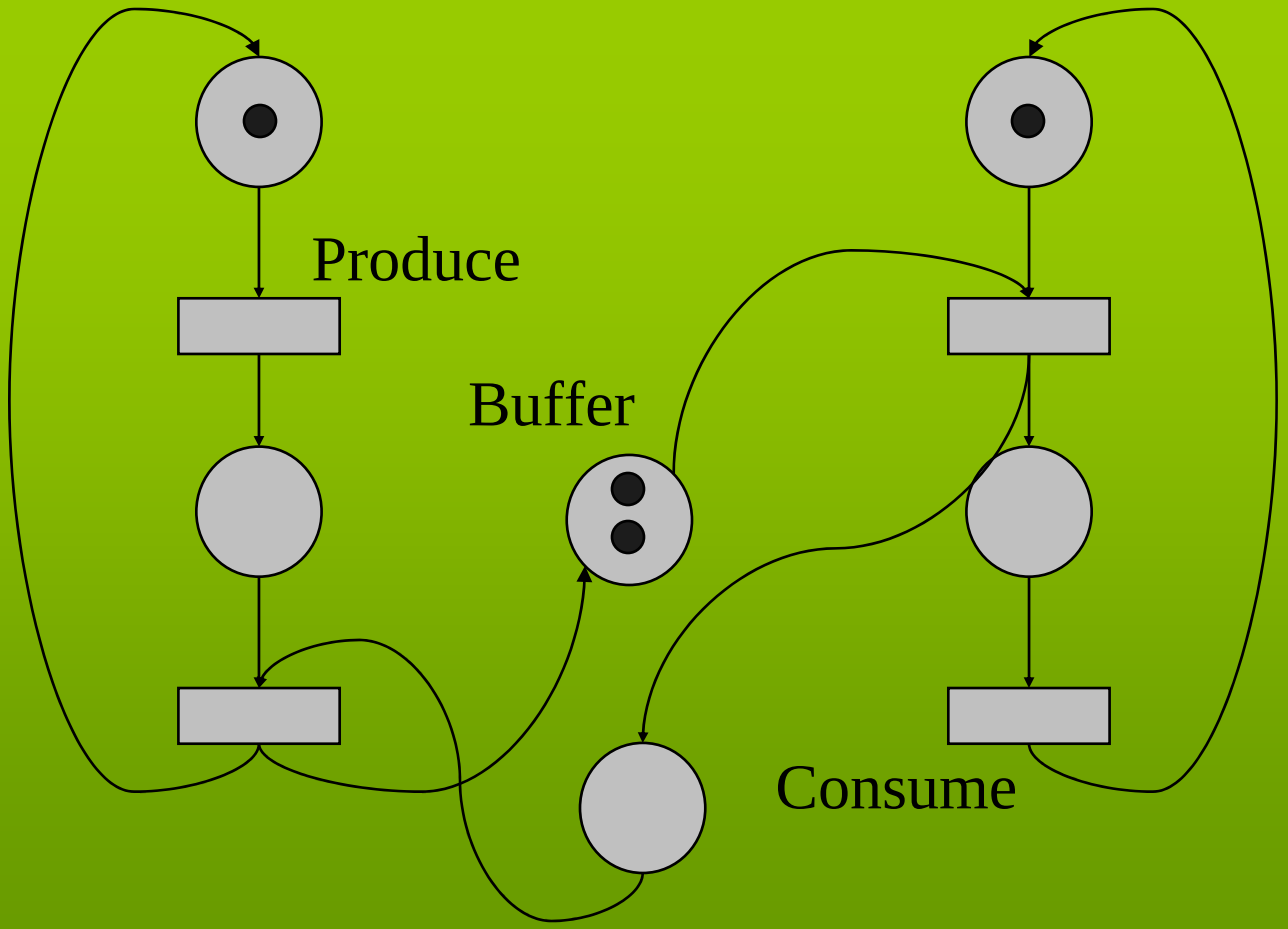
Producer-Consumer Problem



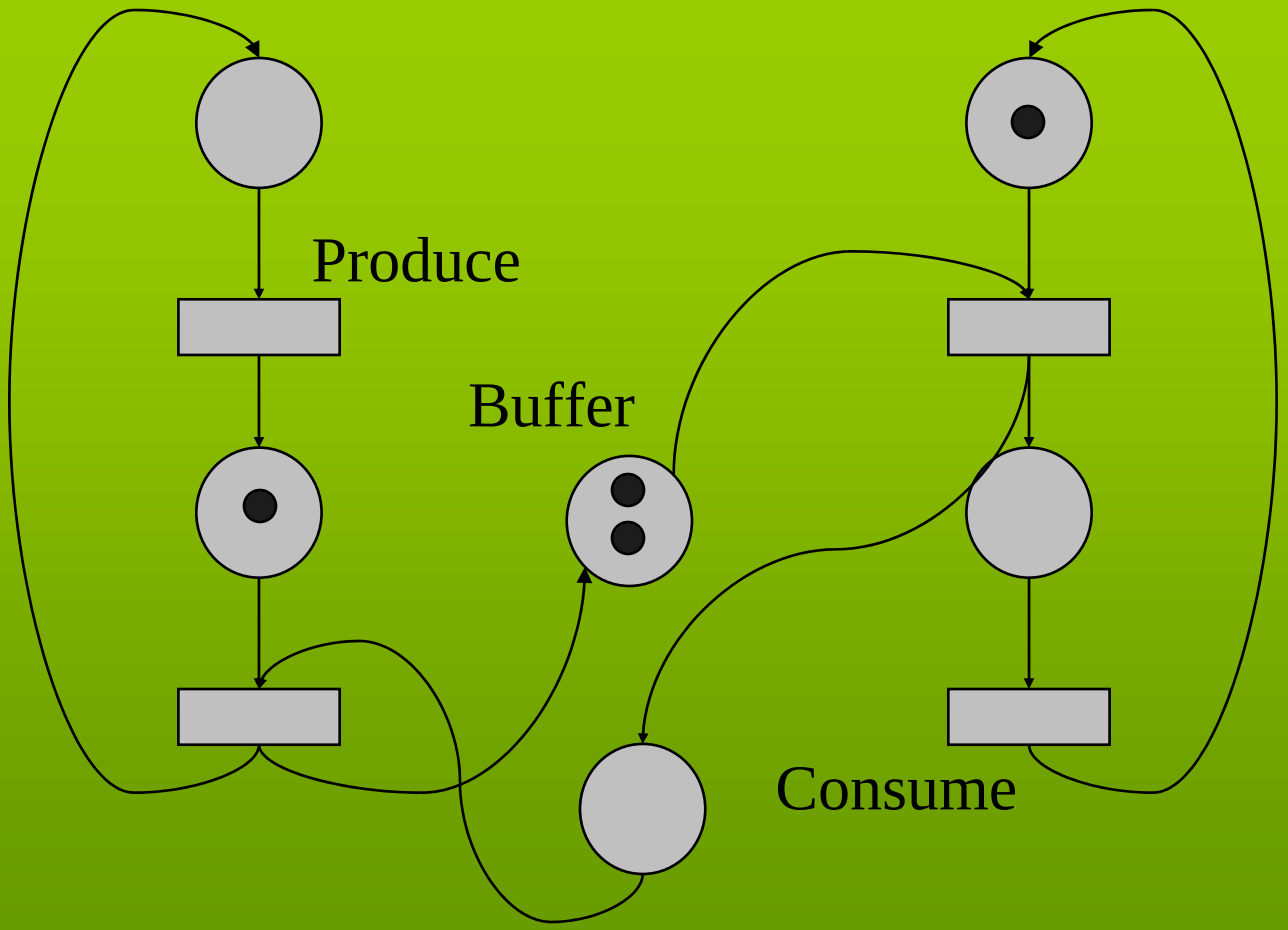
Producer-Consumer Problem



Producer-Consumer Problem



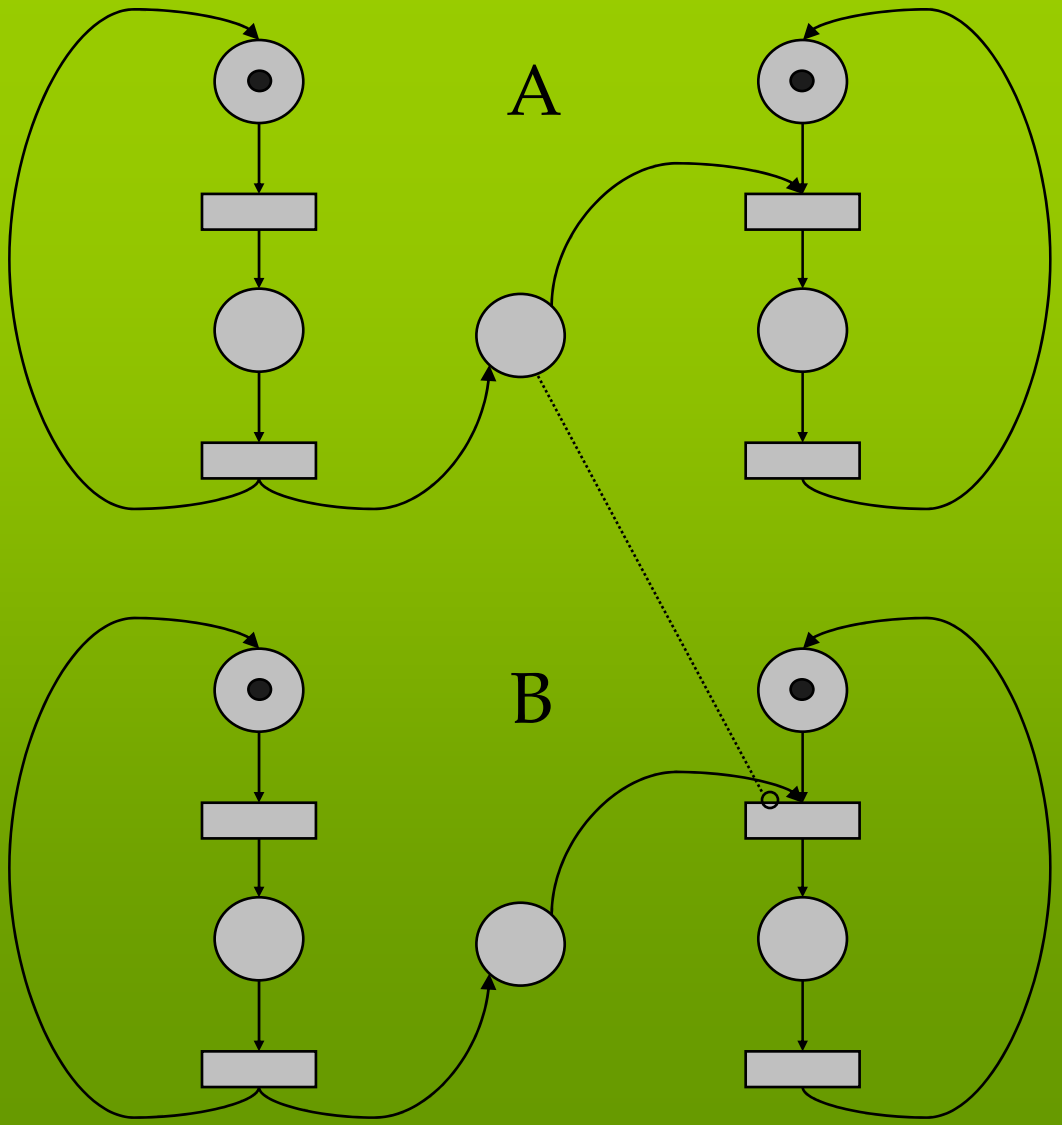
Producer-Consumer Problem



Producer-Consumer with priority

Consumer B can consume only if buffer A is empty

Inhibitor arcs

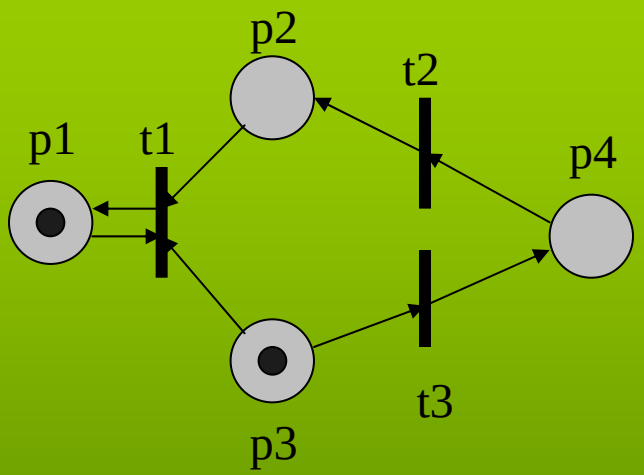


PN properties

- **Behavioral: depend on the initial marking (most interesting)**
  - **Reachability**
  - **Boundedness**
  - **Schedulability**
  - **Liveness**
  - **Conservation**
- **Structural: do not depend on the initial marking (often too restrictive)**
  - **Consistency**
  - **Structural boundedness**

Reachability

- **Marking M is reachable from marking M_0 if there exists a sequence of firings $\sigma = M_0\ t_1\ M_1\ t_2\ M_2 \ldots$ M that transforms M_0 to M.**
- **The reachability problem is decidable.**

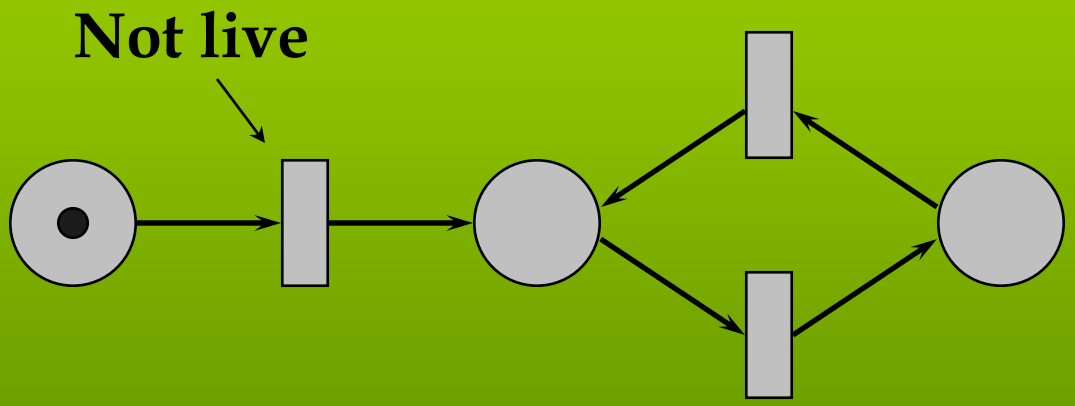


$M_0 = (1,0,1,0)$
$M = (1,1,0,0)$

$M_0 = (1,0,1,0)$
$\downarrow$ t3
$M_1 = (1,0,0,1)$
$\downarrow$ t2
$M = (1,1,0,0)$

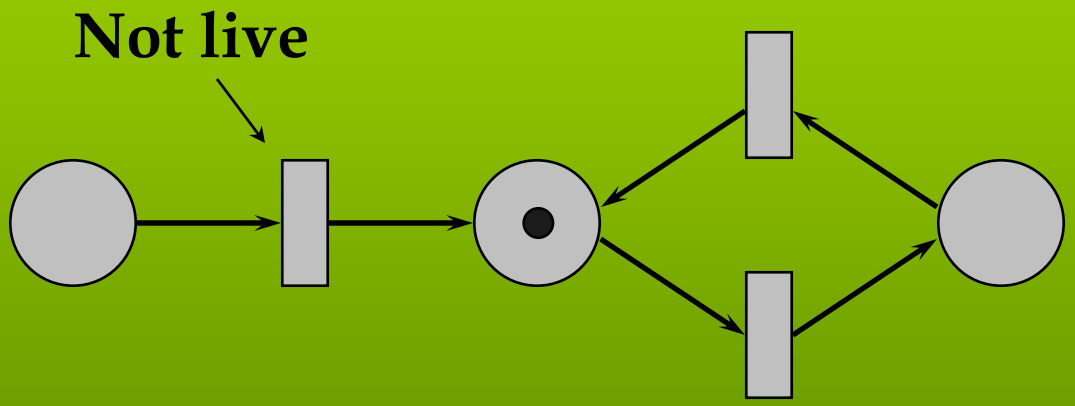
Liveness

- **Liveness: from any marking any transition can become fireable**
  - **Liveness implies deadlock freedom, not viceversa**



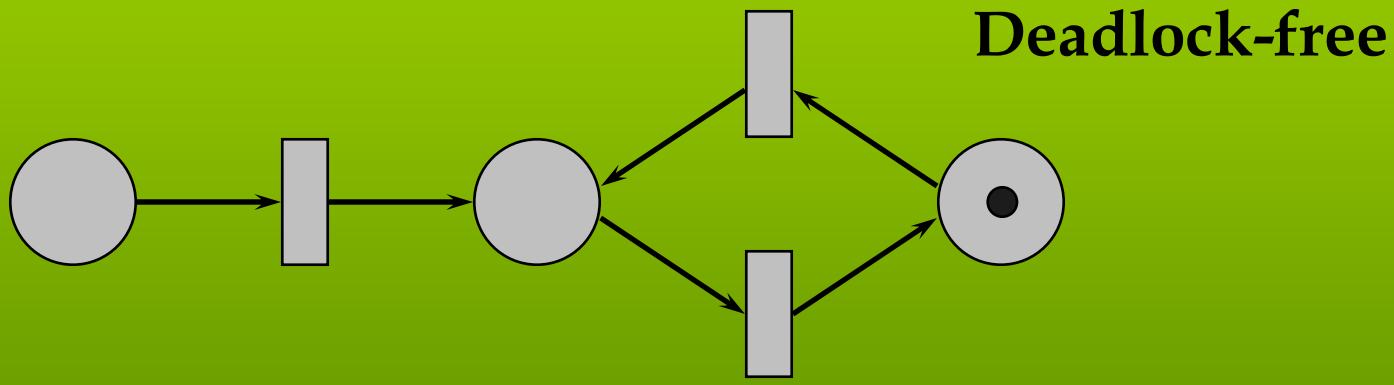
Liveness

- **Liveness: from any marking any transition can become fireable**
  - **Liveness implies deadlock freedom, not viceversa**



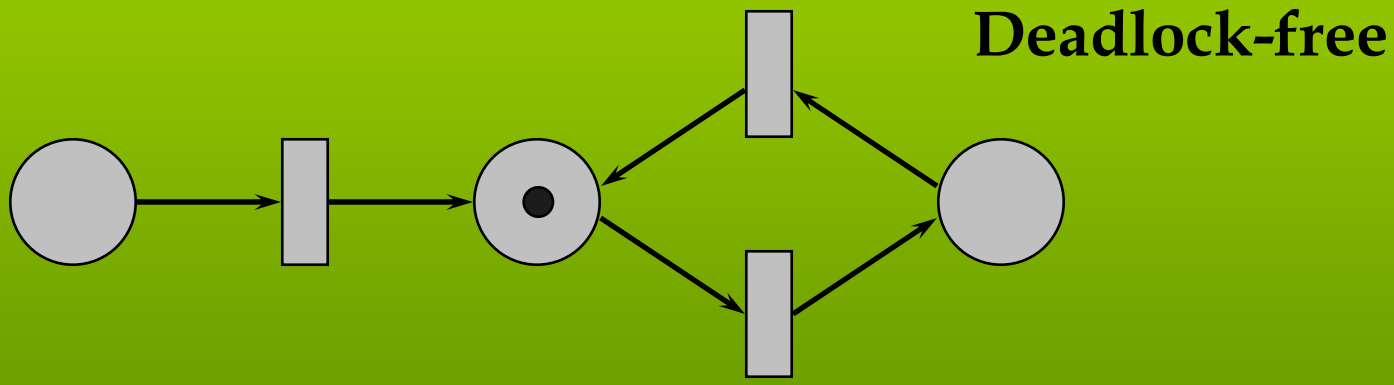
Liveness

- **Liveness: from any marking any transition can become fireable**
  - **Liveness implies deadlock freedom, not viceversa**



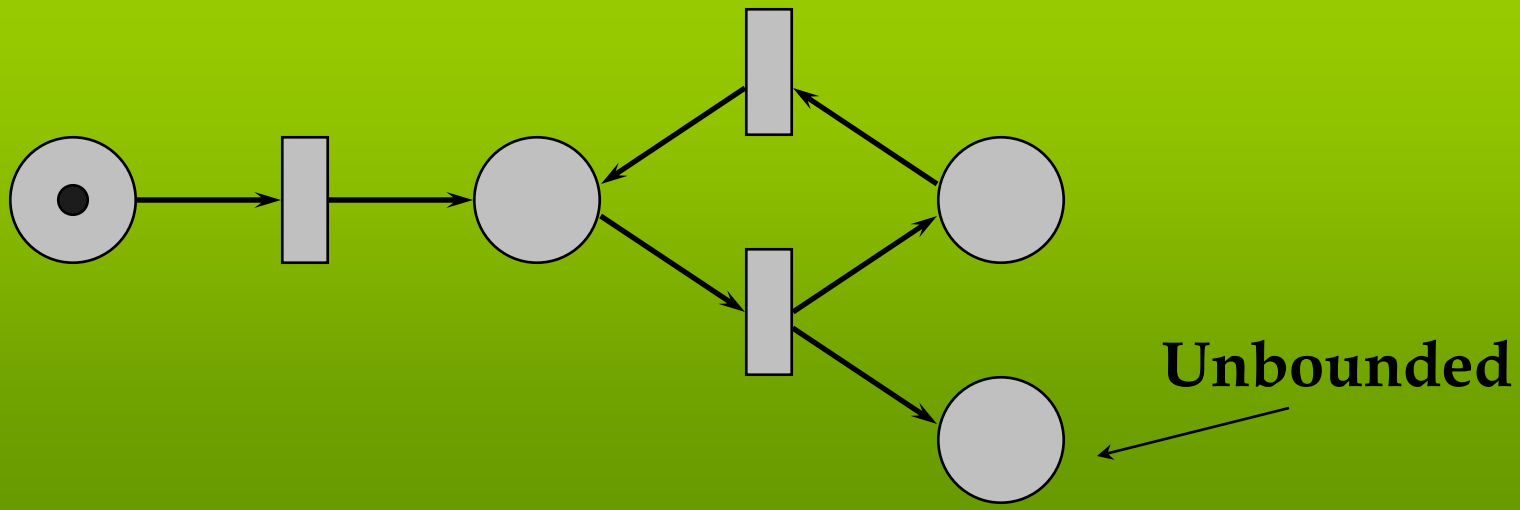
Liveness

- **Liveness: from any marking any transition can become fireable**
  - **Liveness implies deadlock freedom, not viceversa**



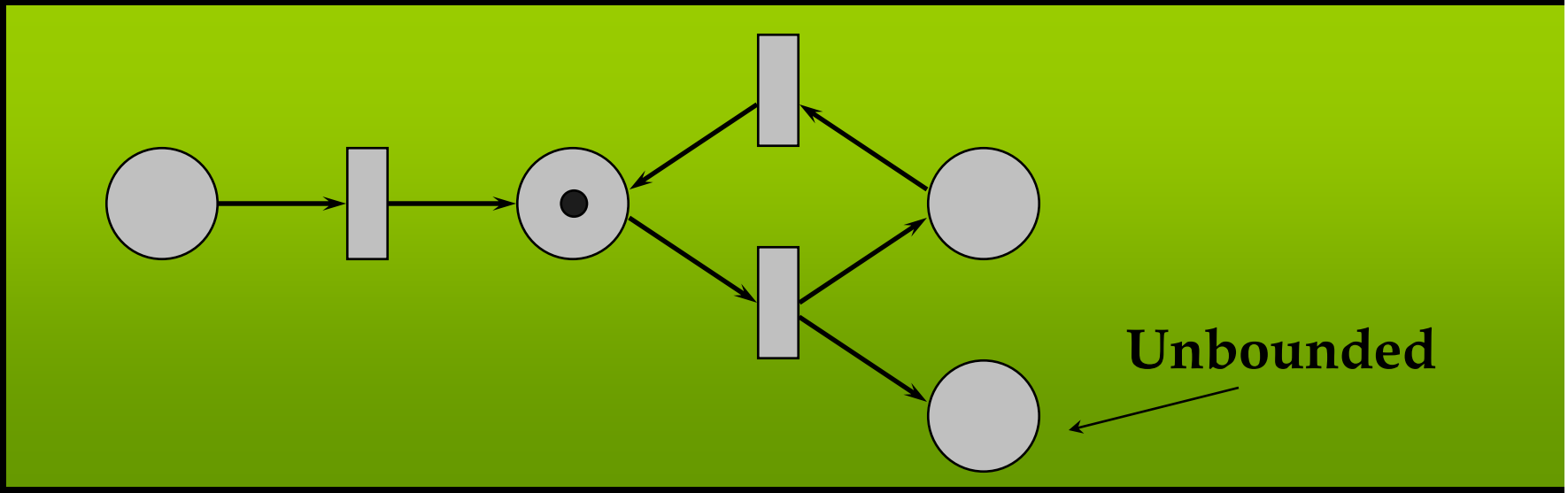
Boundedness

- **Boundedness: the number of tokens in any place cannot grow indefinitely**
  - **(1-bounded also called *safe*)**
  - **Application: places represent buffers and registers (check there is no overflow)**



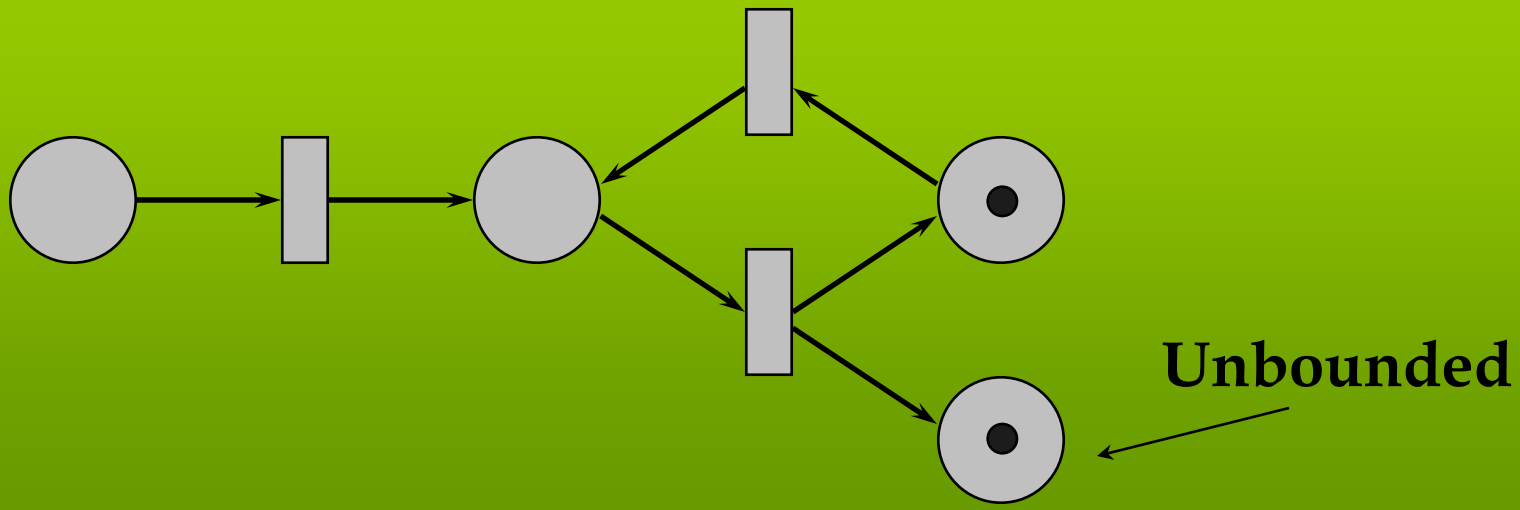
Boundedness

- **Boundedness: the number of tokens in any place cannot grow indefinitely**
  - **(1-bounded also called *safe*)**
  - **Application: places represent buffers and registers (check there is no overflow)**



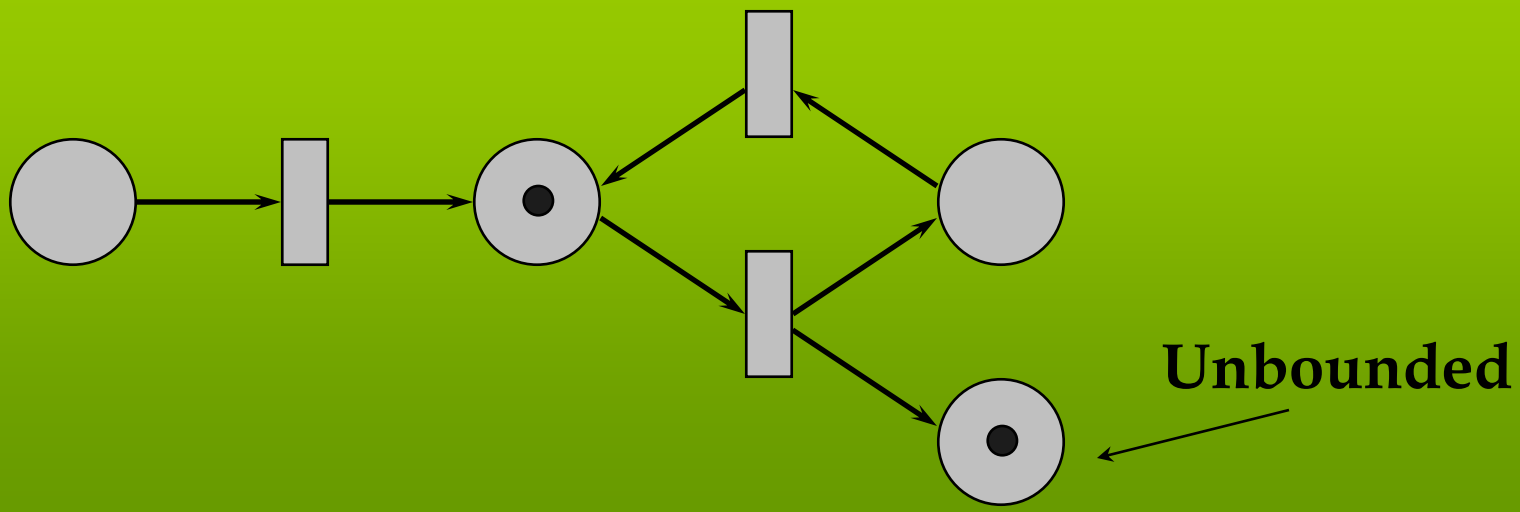
Boundedness

- **Boundedness: the number of tokens in any place cannot grow indefinitely**
  - **(1-bounded also called *safe*)**
  - **Application: places represent buffers and registers (check there is no overflow)**



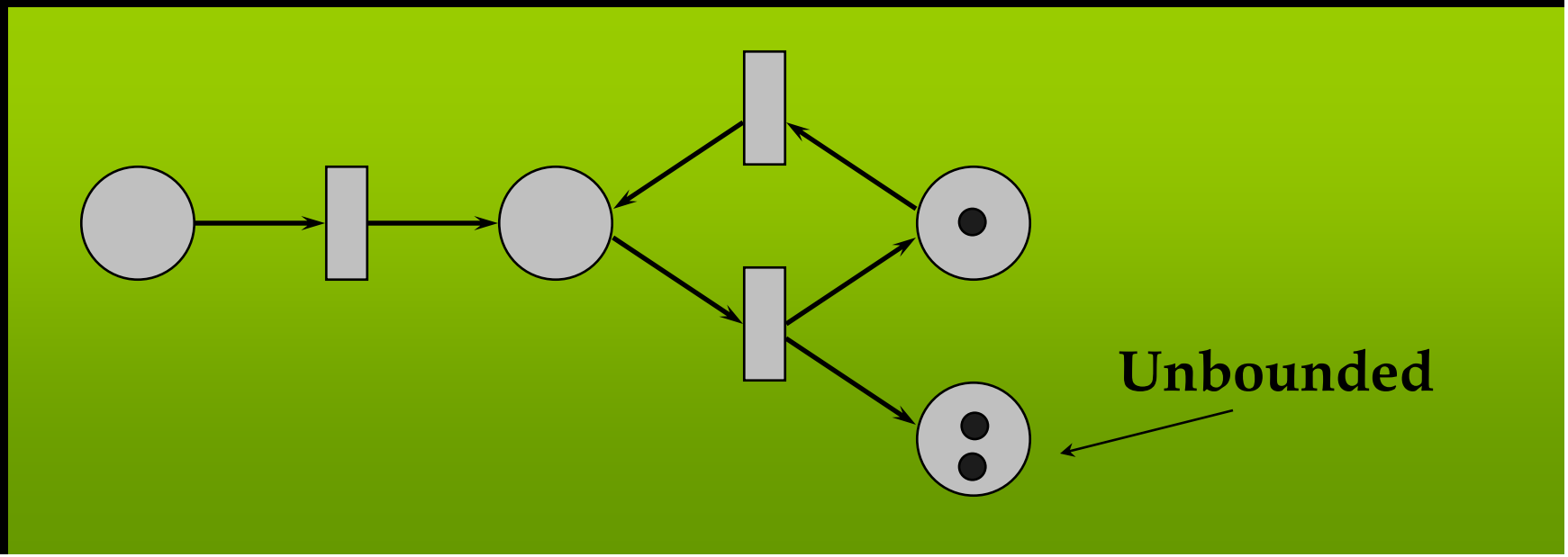
Boundedness

- **Boundedness: the number of tokens in any place cannot grow indefinitely**
  - **(1-bounded also called *safe*)**
  - **Application: places represent buffers and registers (check there is no overflow)**



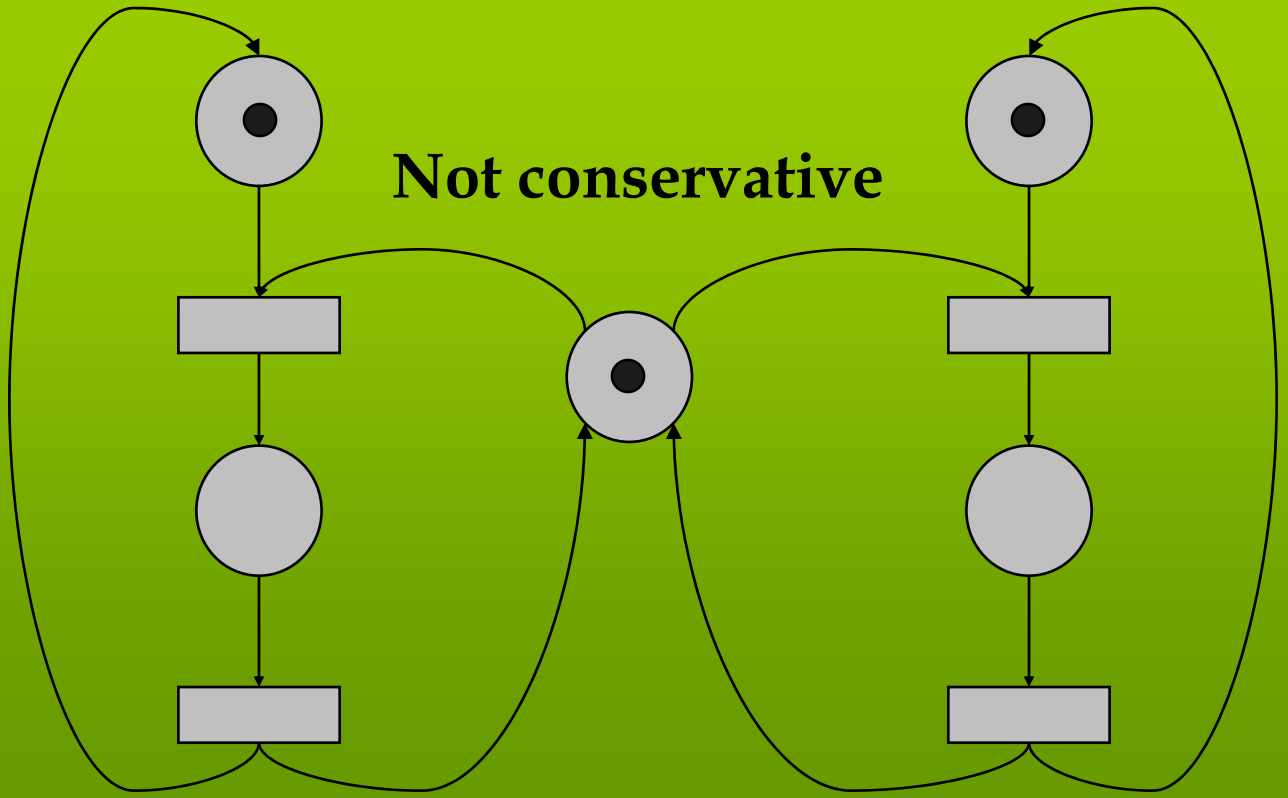
Boundedness

- **Boundedness: the number of tokens in any place cannot grow indefinitely**
  - **(1-bounded also called *safe*)**
  - **Application: places represent buffers and registers (check there is no overflow)**



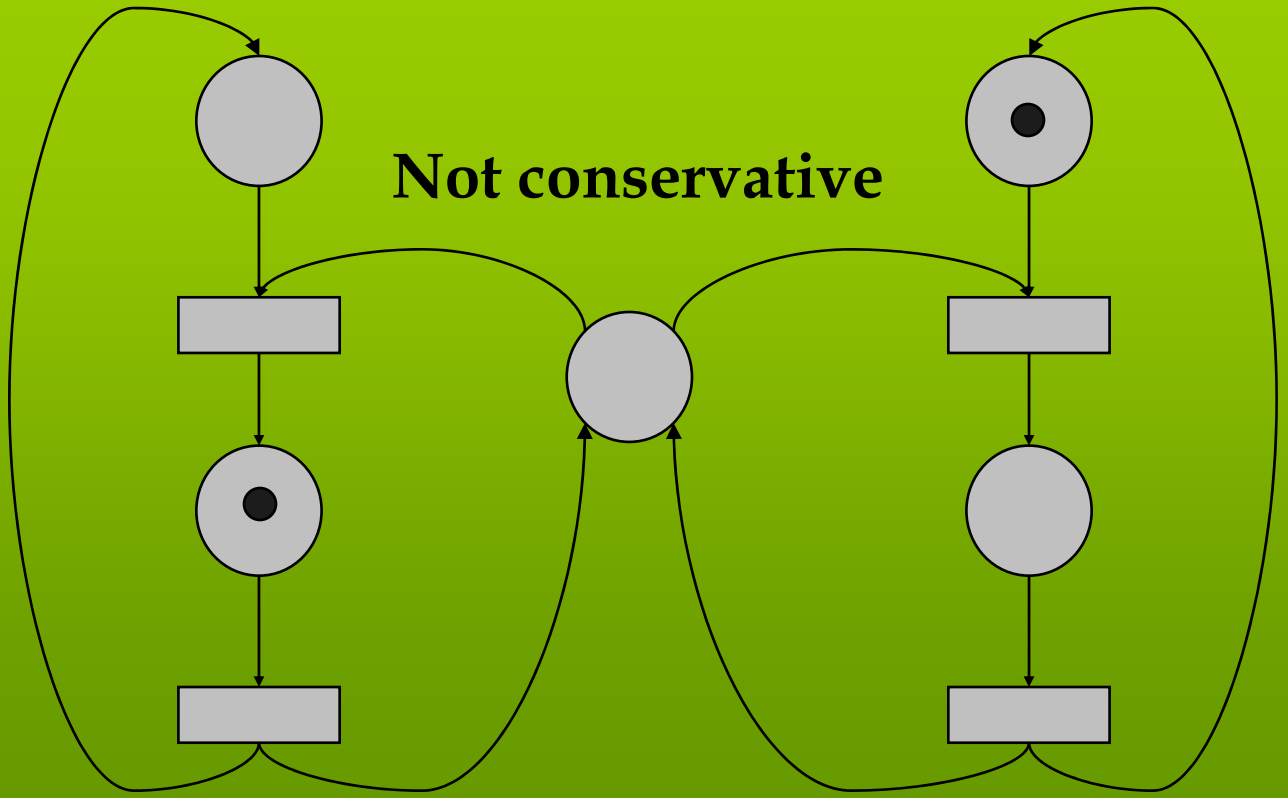
Conservation

- **Conservation: the total number of tokens in the net is constant**



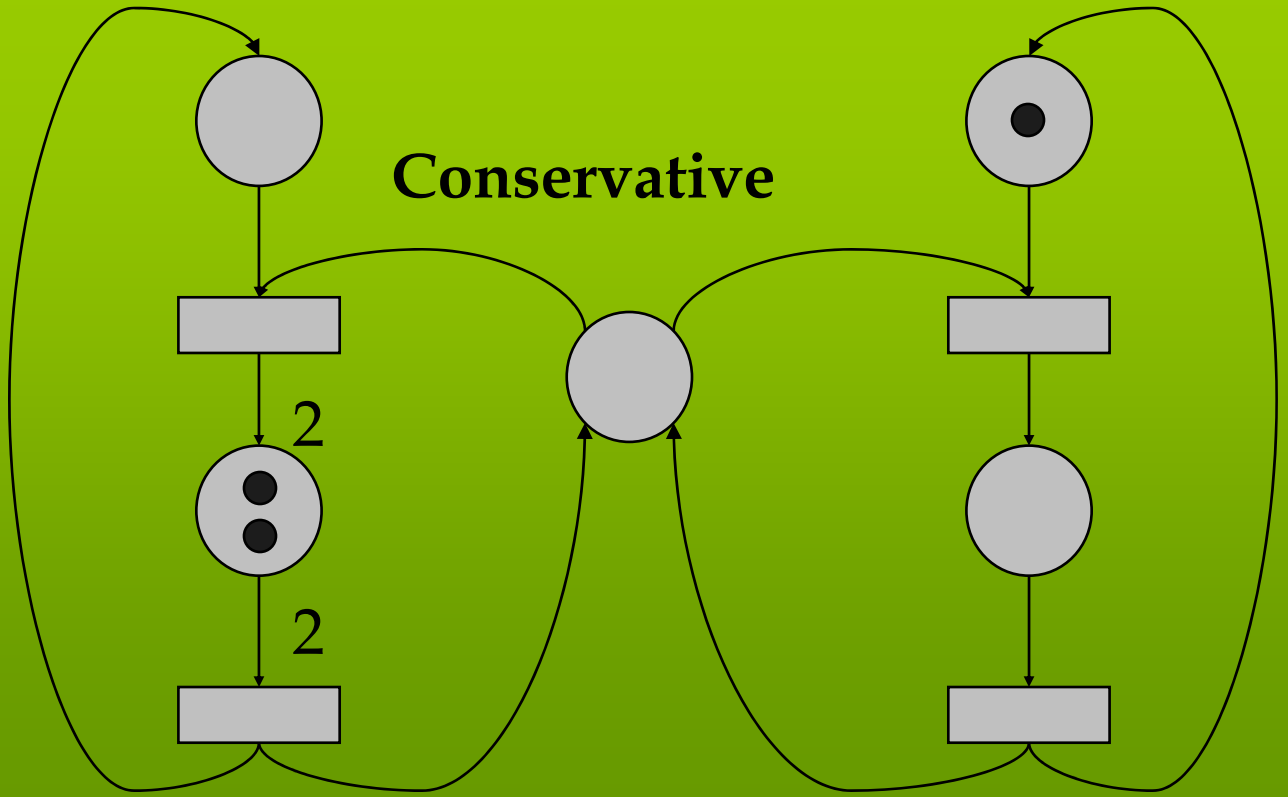
Conservation

- **Conservation: the total number of tokens in the net is constant**



Conservation

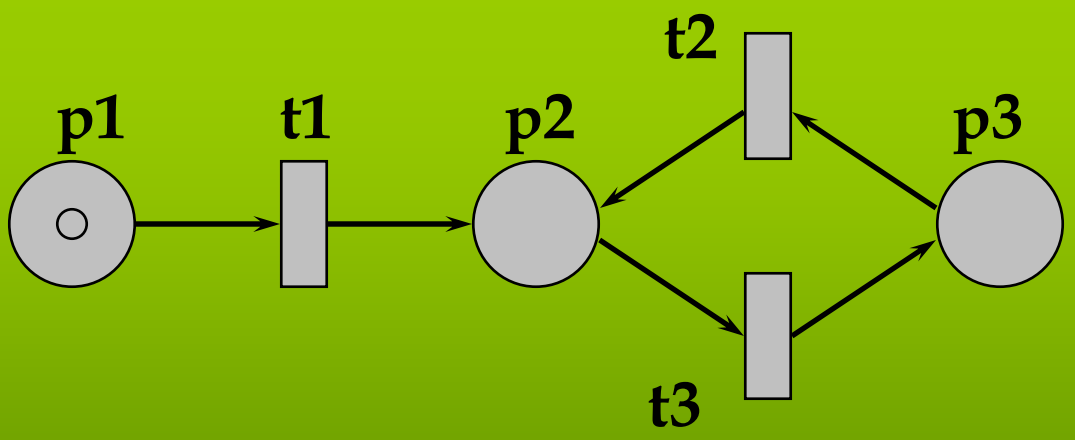
- **Conservation: the total number of tokens in the net is constant**



Analysis techniques

- **Structural analysis techniques**
  - **Incidence matrix**
  - **T- and S- Invariants**
- **State Space Analysis techniques**
  - **Coverability Tree**
  - **Reachability Graph**

Incidence Matrix



$$A = \begin{array}{c|ccc} & t1 & t2 & t3 \\ \hline p1 & -1 & 0 & 0 \\ p2 & 1 & 1 & -1 \\ p3 & 0 & -1 & 1 \end{array}$$

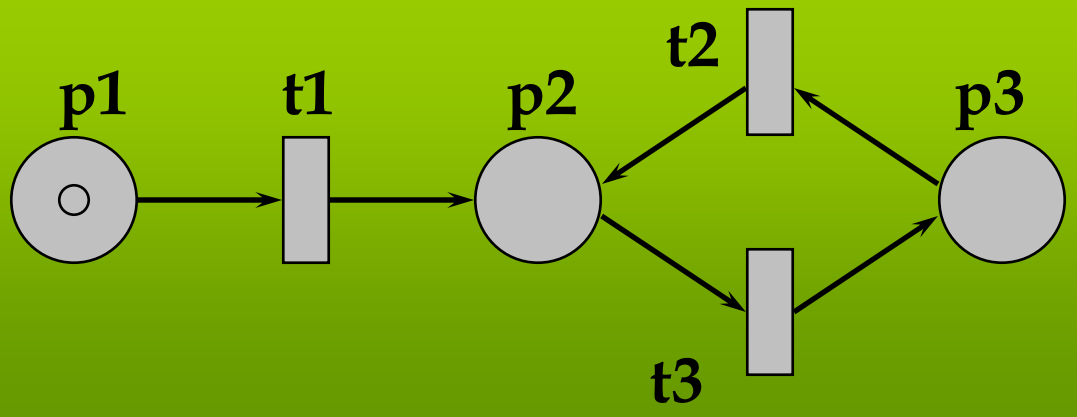
- **Necessary condition for marking M to be reachable from initial marking M_0:**

  there exists firing vector v s.t.:

  $$M = M_0 + A\,v$$

State equations

- E.g. reachability of M = $|0\ 0\ 1|^T$ from $M_0 = |1\ 0\ 0|^T$

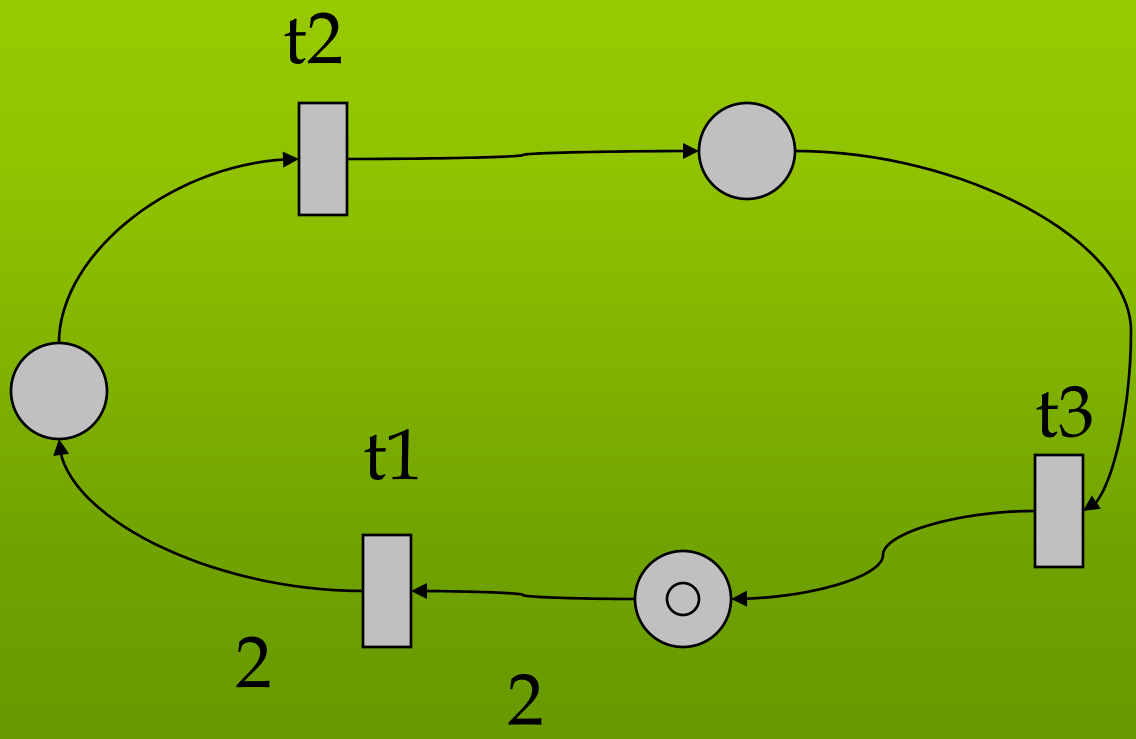


$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$v_1 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

but also $v_2 = |\ 1\ 1\ 2\ |^T$ or any $v_k = |\ 1\ (k)\ (k+1)\ |^T$

Necessary Condition only



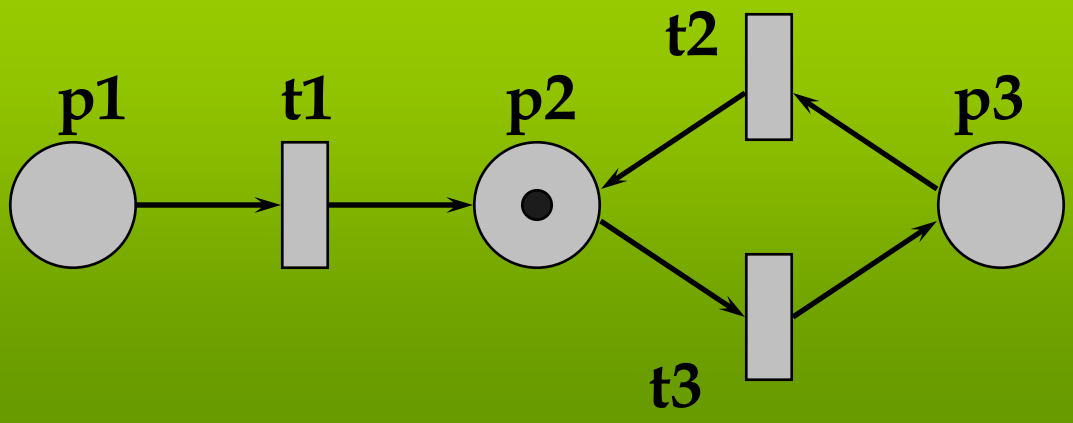
Deadlock!!

State equations and invariants

- **Solutions of Ax = 0 (in $M = M_0 + Ax$, $M = M_0$)**

  T-invariants

  - **sequences of transitions that (if fireable) bring back to original marking**
  - **periodic schedule in SDF**
  - **e.g. $x = |\ 0\ 1\ 1\ |^T$**



$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

Application of T-invariants

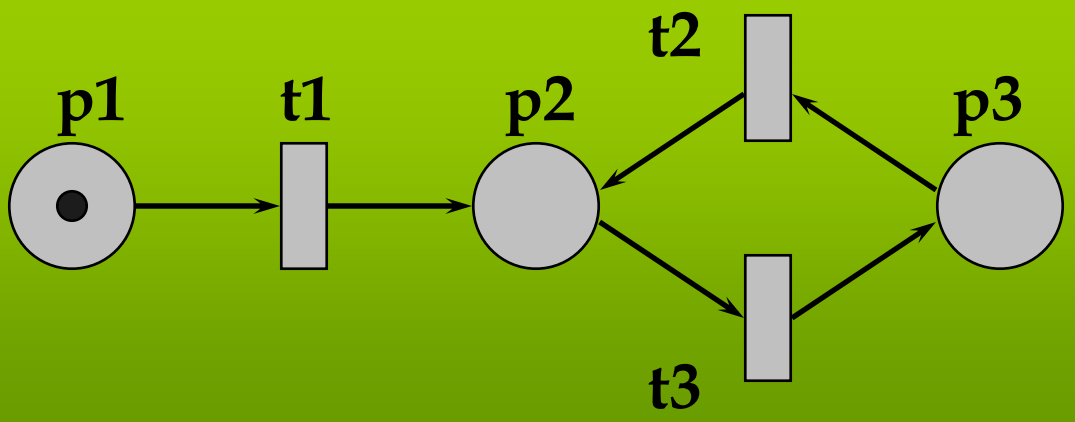
- **Scheduling**
  - **Cyclic schedules: need to return to the initial state**

State equations and invariants

- **Solutions of yA = 0**

  S-invariants

  - **sets of places whose weighted total token count does not change after the firing of any transition (y M = y M')**
  - **e.g. $y = | 1\ 1\ 1 |^T$**



$$A^T = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

Application of S-invariants

- **Structural Boundedness: bounded for any finite initial marking M_0**
- **Existence of a positive S-invariant is CS for structural boundedness**
  - **initial marking is finite**
  - **weighted token count does not change**

Summary of algebraic methods

- **Extremely efficient**

  (polynomial in the size of the net)
- **Generally provide only necessary or sufficient information**
- **Excellent for ruling out some deadlocks or otherwise dangerous conditions**
- **Can be used to infer structural boundedness**

Coverability Tree

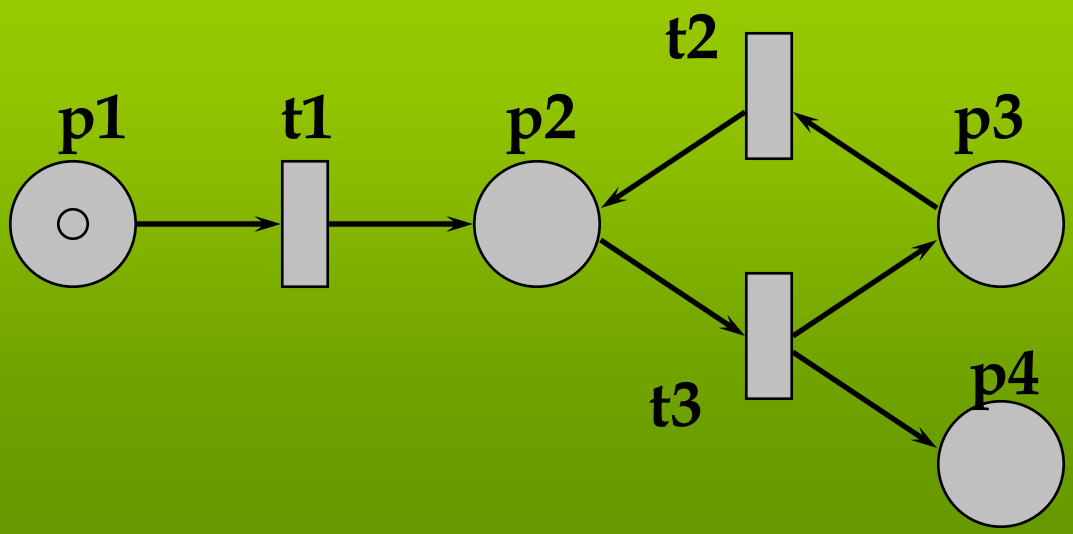
- **Build a (finite) tree representation of the markings**

Karp-Miller algorithm

- **Label initial marking M0 as the root of the tree and tag it as *new***
- **While new markings exist do:**
  - **select a new marking M**
  - **if M is identical to a marking on the path from the root to M, then tag M as *old* and go to another new marking**
  - **if no transitions are enabled at M, tag M *dead-end***
  - **while there exist enabled transitions at M do:**
    - **obtain the marking M' that results from firing t at M**
    - **on the path from the root to M if there exists a marking M'' such that M'(p)>=M''(p) for each place p and M' is different from M'', then replace M'(p) by ω for each p such that M'(p) >M''(p)**
    - **introduce M' as a node, draw an arc with label t from M to M' and tag M' as *new*.**

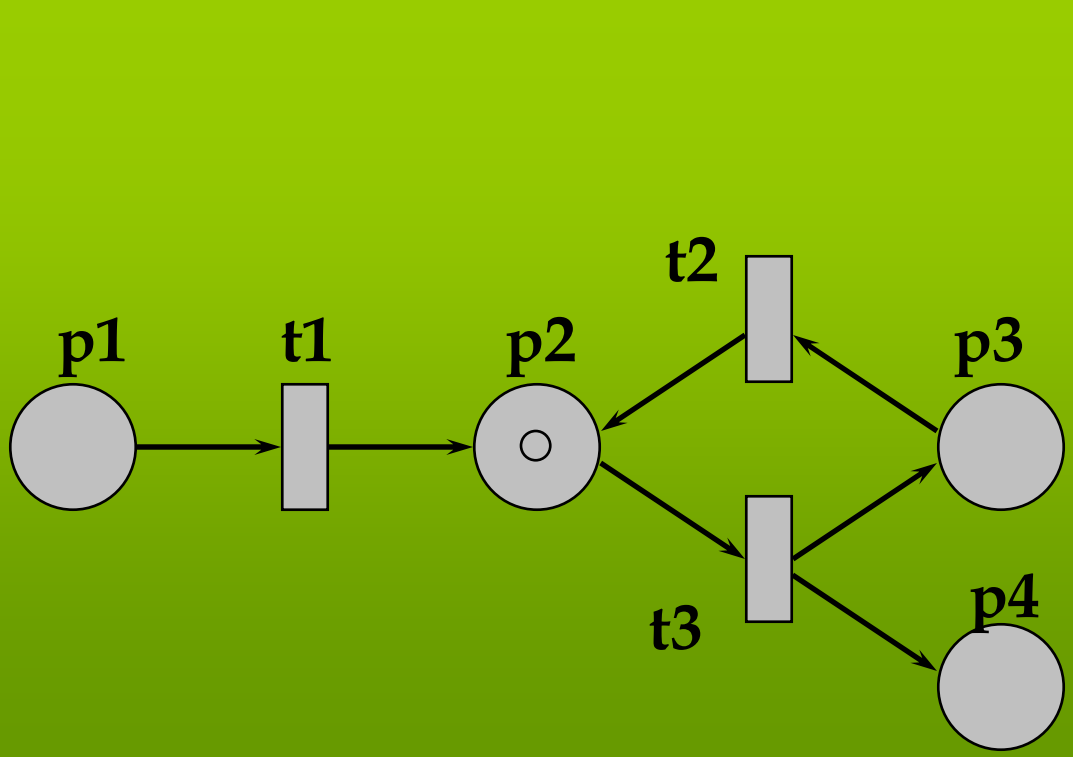
Coverability Tree

- **Boundedness is decidable with *coverability tree***



Coverability Tree

- **Boundedness is decidable with *coverability tree***



1000

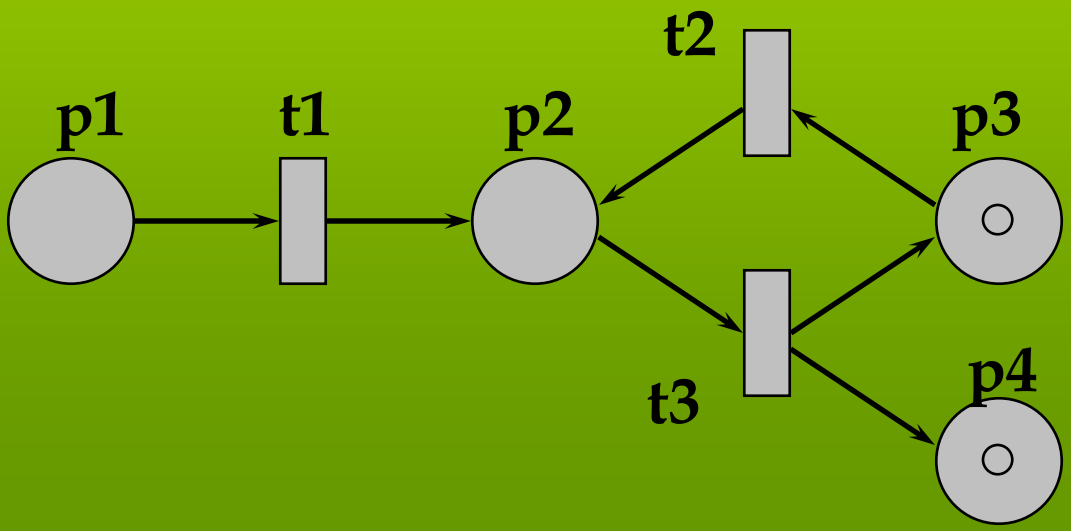
↓ t1

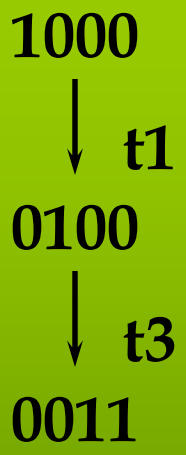
0100

Coverability Tree

- **Boundedness is decidable**

  with *coverability tree*

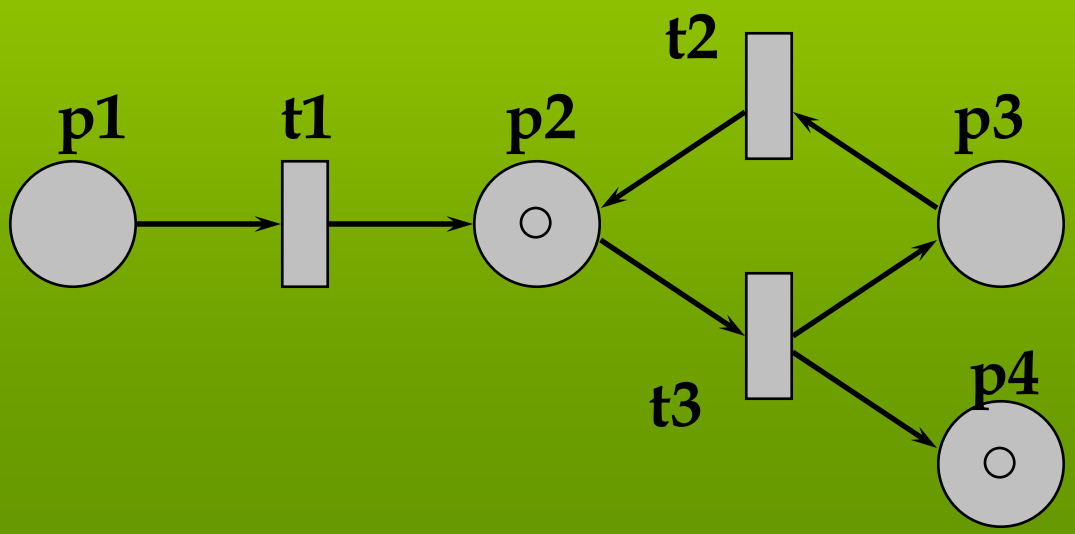


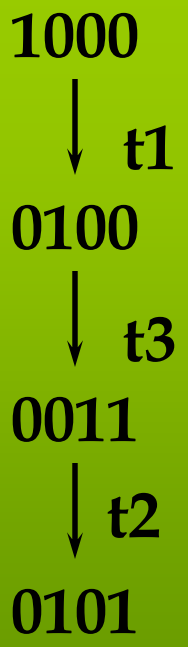


Coverability Tree

- **Boundedness is decidable**

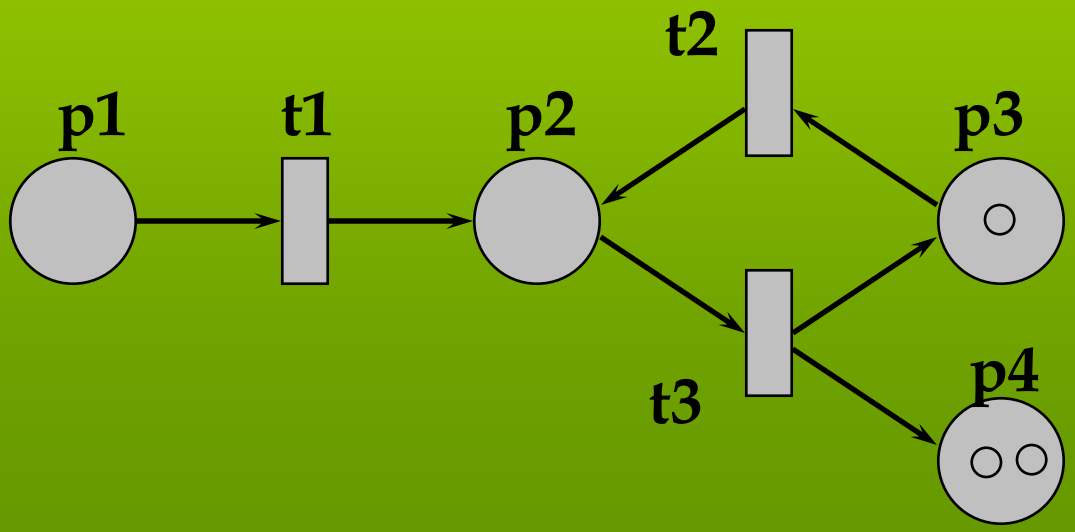
  with *coverability tree*





Coverability Tree

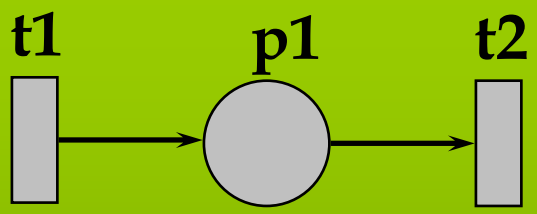
- **Boundedness is decidable with *coverability tree***

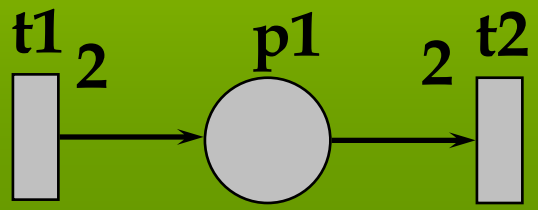


1000
↓ t1
0100
↓ t3
0011
↓ t2
010ω

Coverability Tree

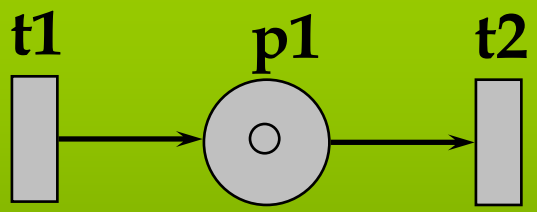
- **Is (1) reachable from (0)?**

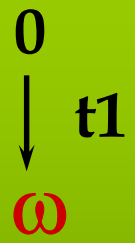


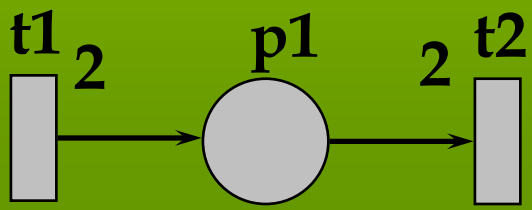


Coverability Tree

- **Is (1) reachable from (0)?**

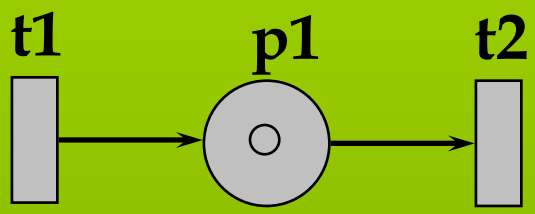


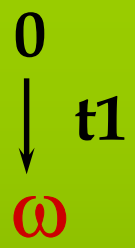




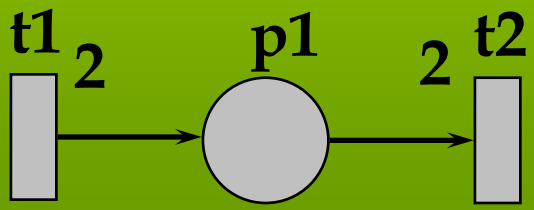
Coverability Tree

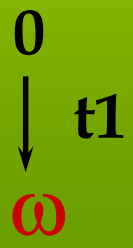
- **Is (1) reachable from (0)?**





$(0) \Longrightarrow (1) \Longrightarrow (2) \ldots$

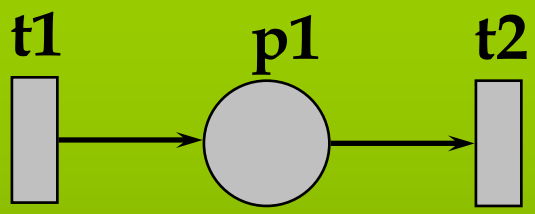


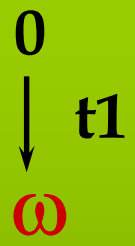


$(0) \longrightarrow (2) \longrightarrow (0) \ldots$

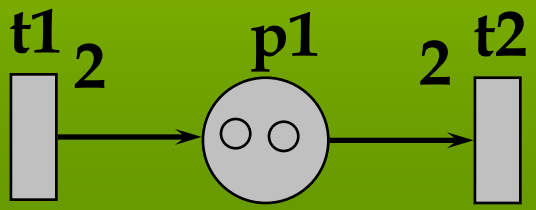
Coverability Tree

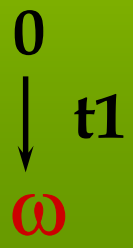
- **Is (1) reachable from (0)?**





$(0) \longrightarrow (1) \longrightarrow (2) \ldots$

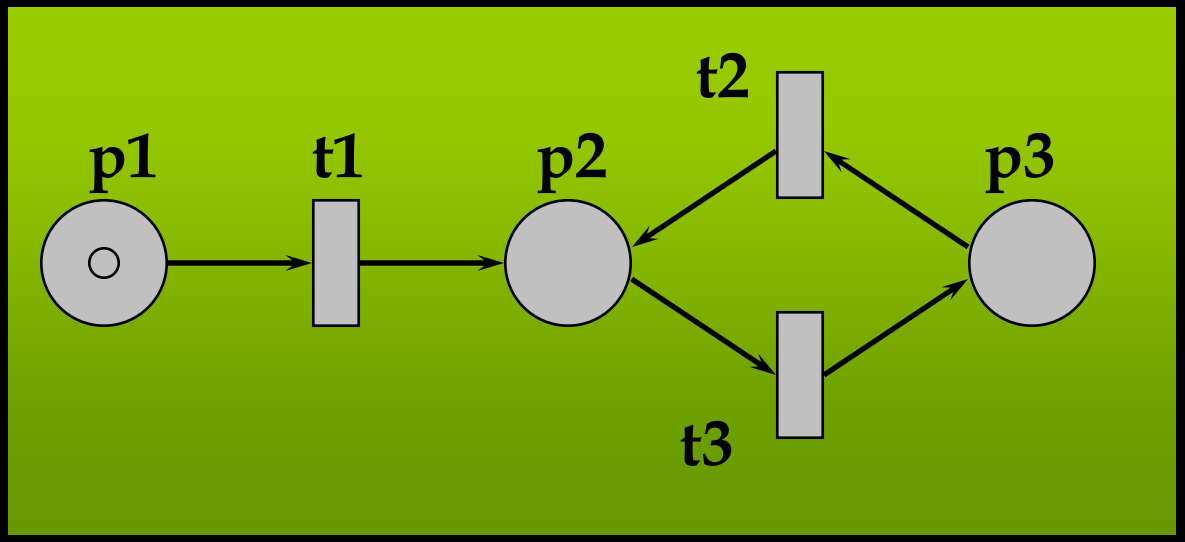




$(0) \longrightarrow (2) \longrightarrow (0) \ldots$

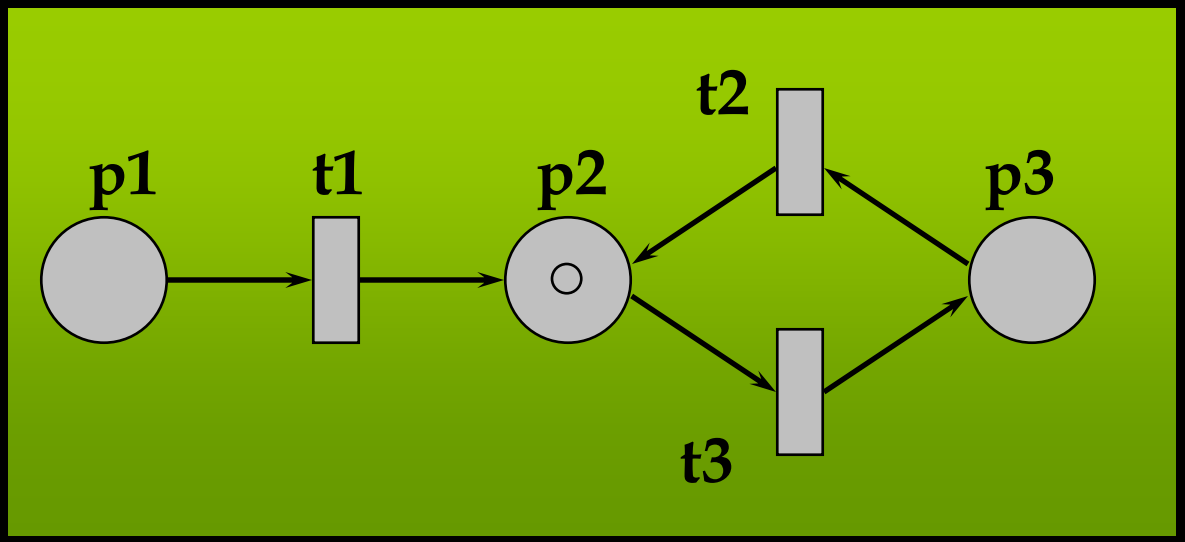
- **Cannot solve the reachability problem**

Reachability graph



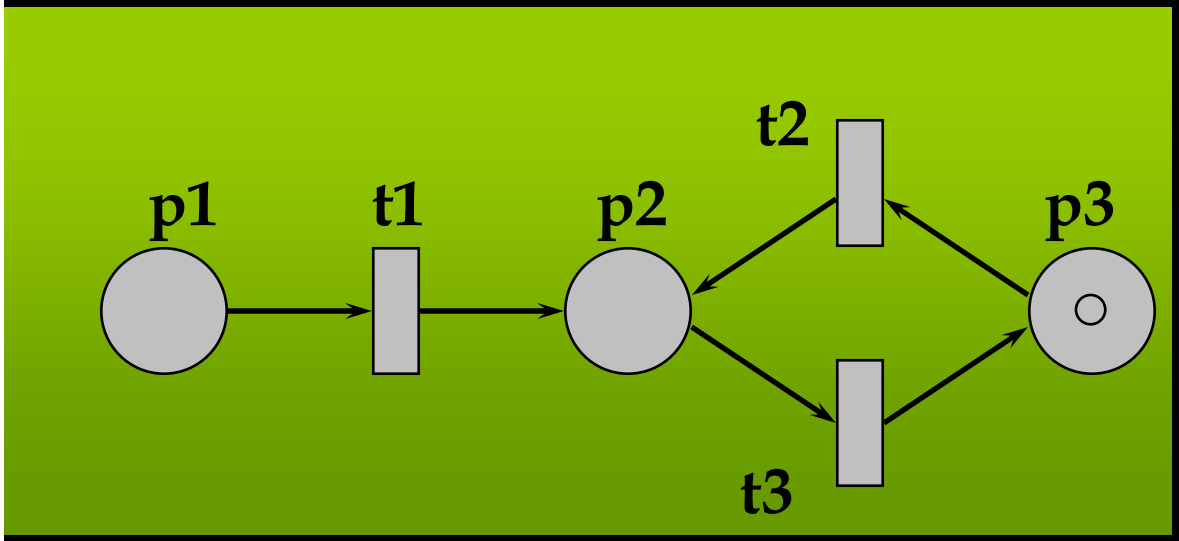
- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

Reachability graph



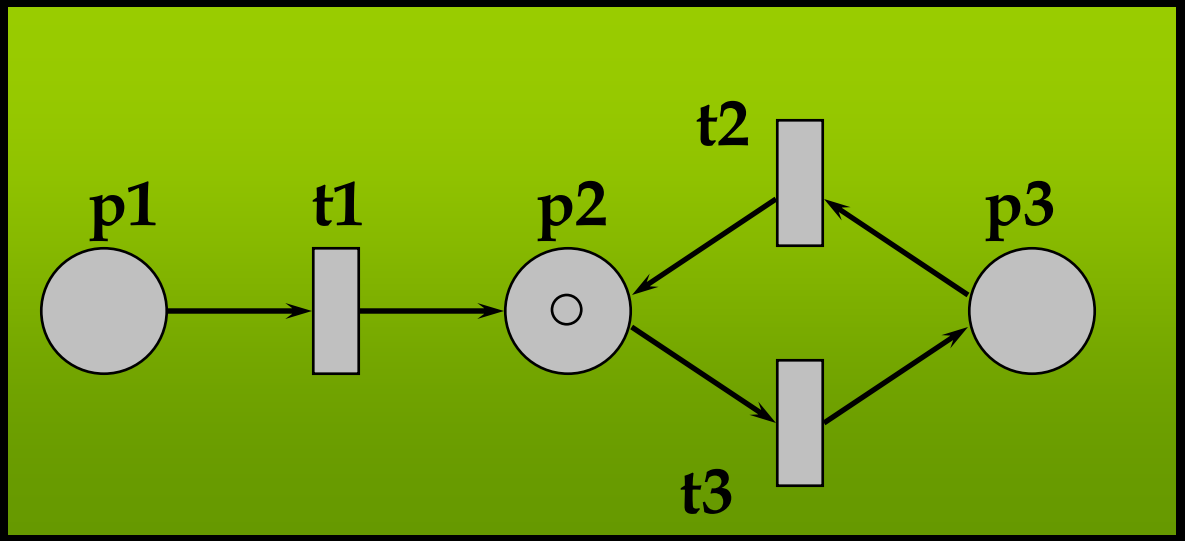
- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

Reachability graph



- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

Reachability graph

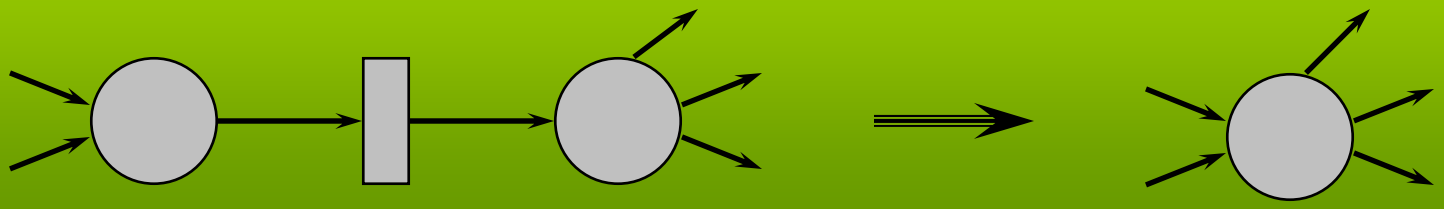


- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

Subclasses of Petri nets

- **Reachability analysis is too expensive**
- **State equations give only partial information**
- **Some properties are preserved by reduction rules**

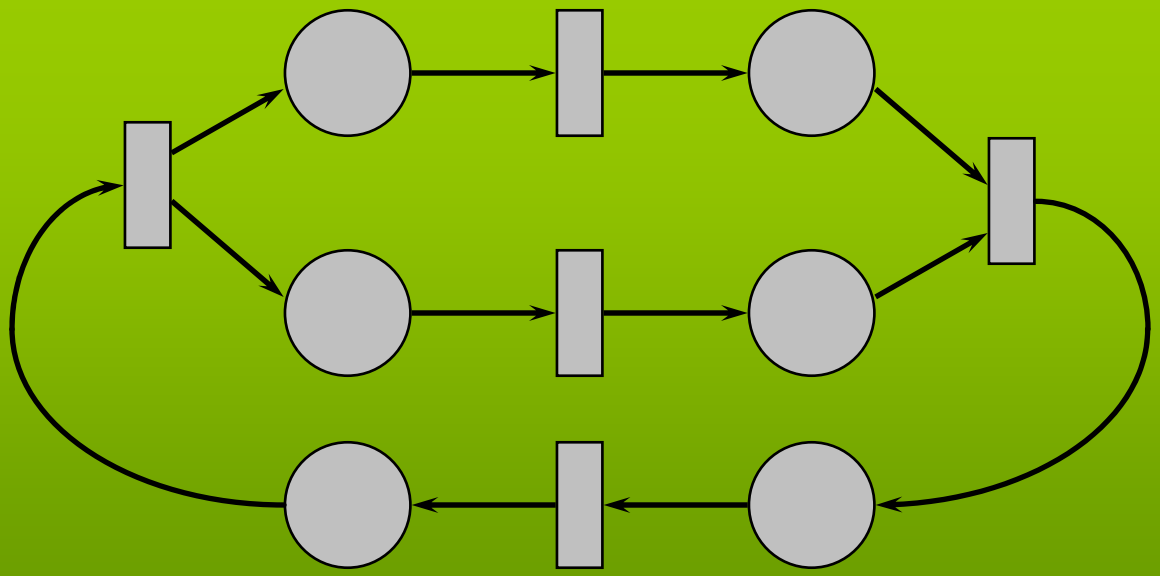
  e.g. for liveness and safeness



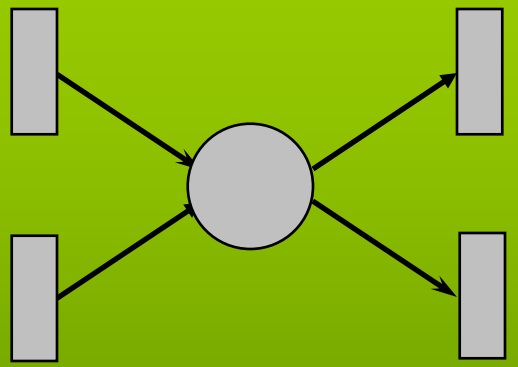
- **Even reduction rules only work in some cases**
- **Must restrict class in order to prove stronger results**

Marked Graphs

- **Every place has at most 1 predecessor and 1 successor transition**
- **Models only causality and concurrency (no conflict)**



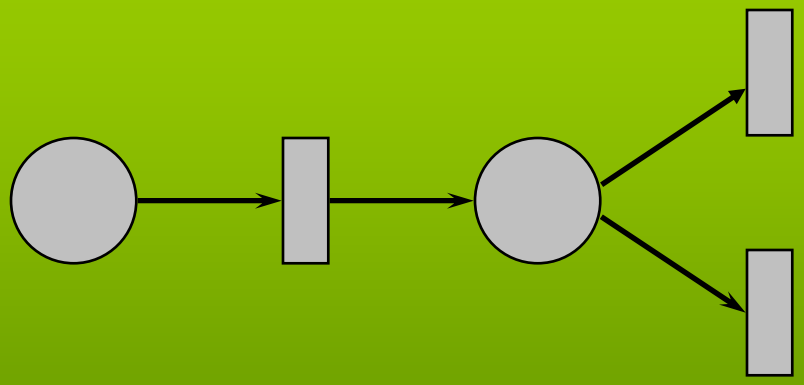
YES



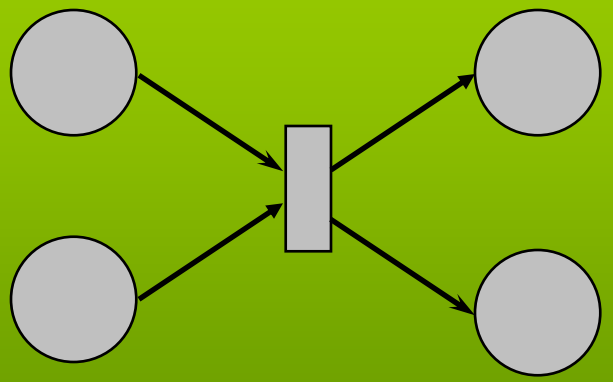
NO

State Machines

- **Every transition has at most 1 predecessor and 1 successor place**
- **Models only causality and conflict**
  - **(no concurrency, no synchronization of parallel activities)**

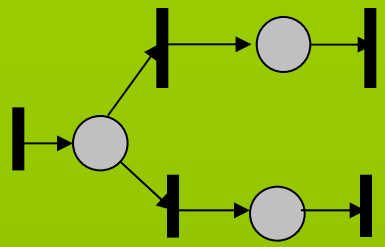


YES



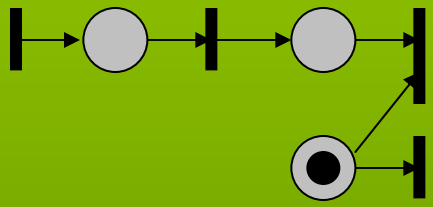
NO

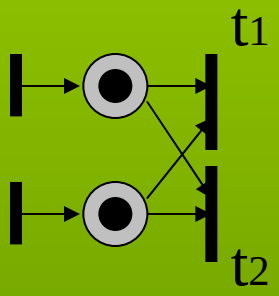
Free-Choice Petri Nets (FCPN)



every transition after choice has exactly 1 predecessor place

Free-Choice (FC)





Confusion (not-Free-Choice) **Extended Free-Choice**

Free-Choice: the outcome of a choice depends on the value of a token (abstracted non-deterministically) rather than on its arrival time.

Free-Choice nets

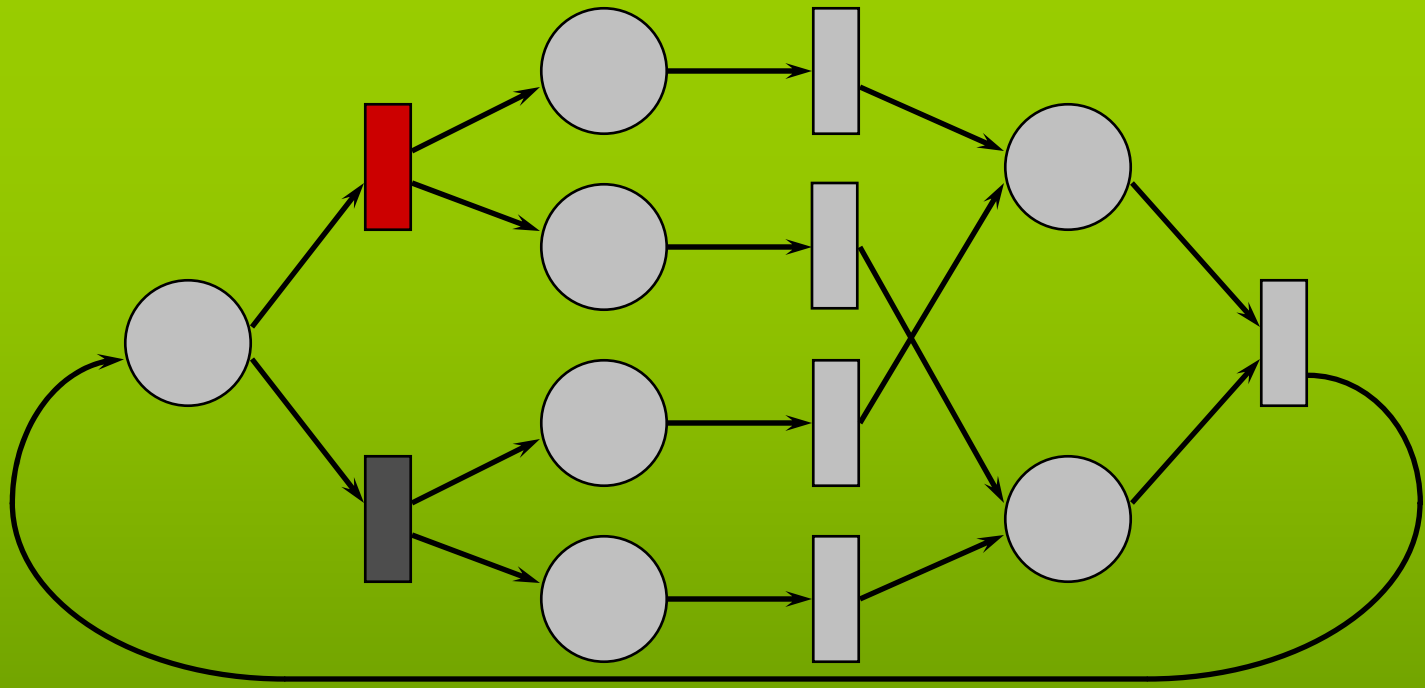
- **Introduced by Hack (‘72)**
- **Extensively studied by Best (‘86) and Desel and Esparza (‘95)**
- **Can express concurrency, causality and choice without confusion**
- **Very strong structural theory**
  - **necessary and sufficient conditions for liveness and safeness, based on decomposition**
  - **exploits duality between MG and SM**

MG (& SM) decomposition

- An **Allocation** is a control function that chooses which transition fires among several conflicting ones (A: P T).
- Eliminate the subnet that would be inactive if we were to use the allocation...
- **Reduction Algorithm**
  - Delete all unallocated transitions
  - Delete all places that have all input transitions already deleted
  - Delete all transitions that have at least one input place already deleted
- Obtain a **Reduction** (one for each allocation) that is a conflict free subnet

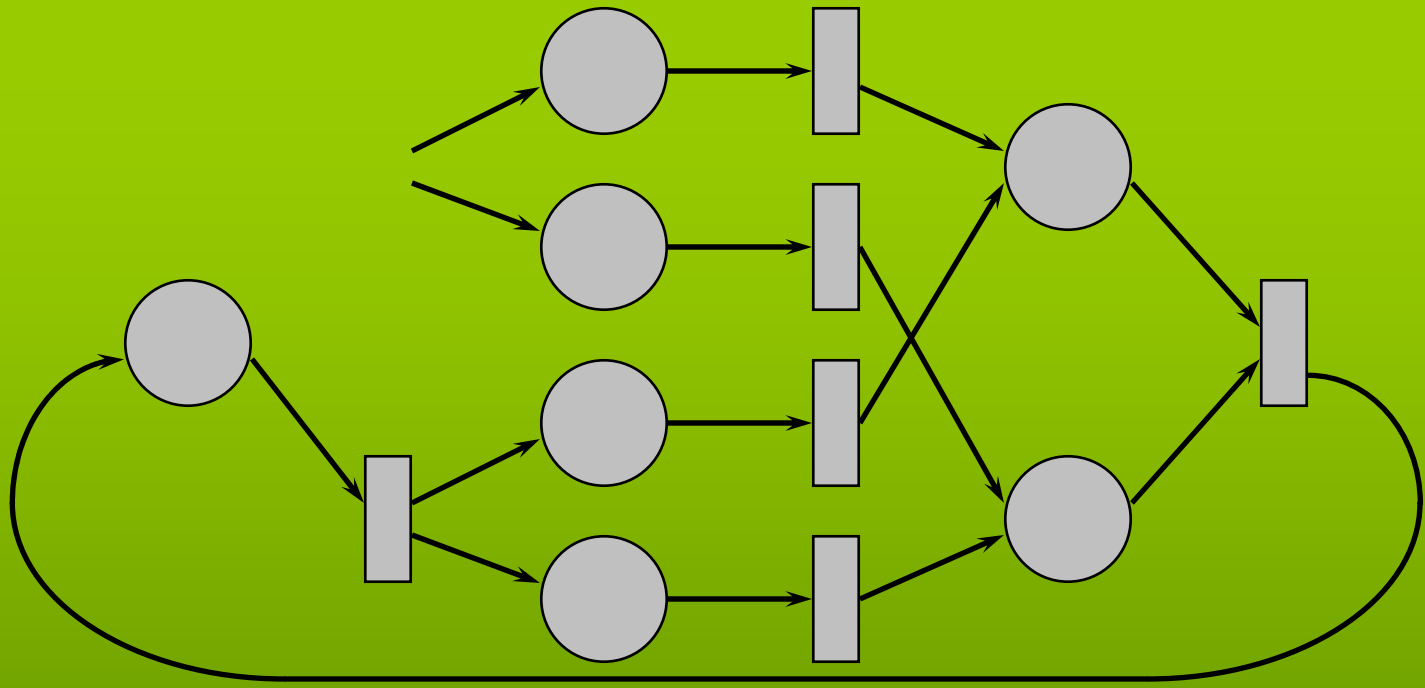
MG reduction and cover

- **Choose one successor for each conflicting place:**



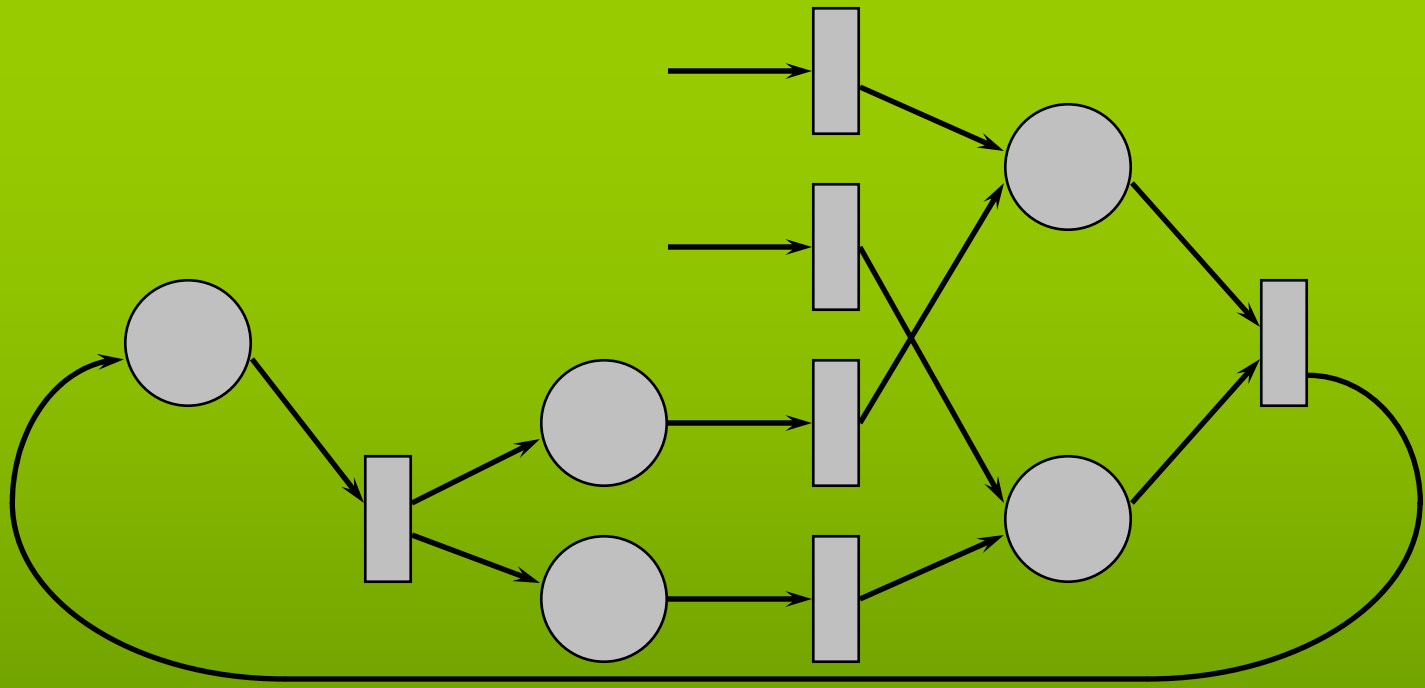
MG reduction and cover

- **Choose one successor for each conflicting place:**



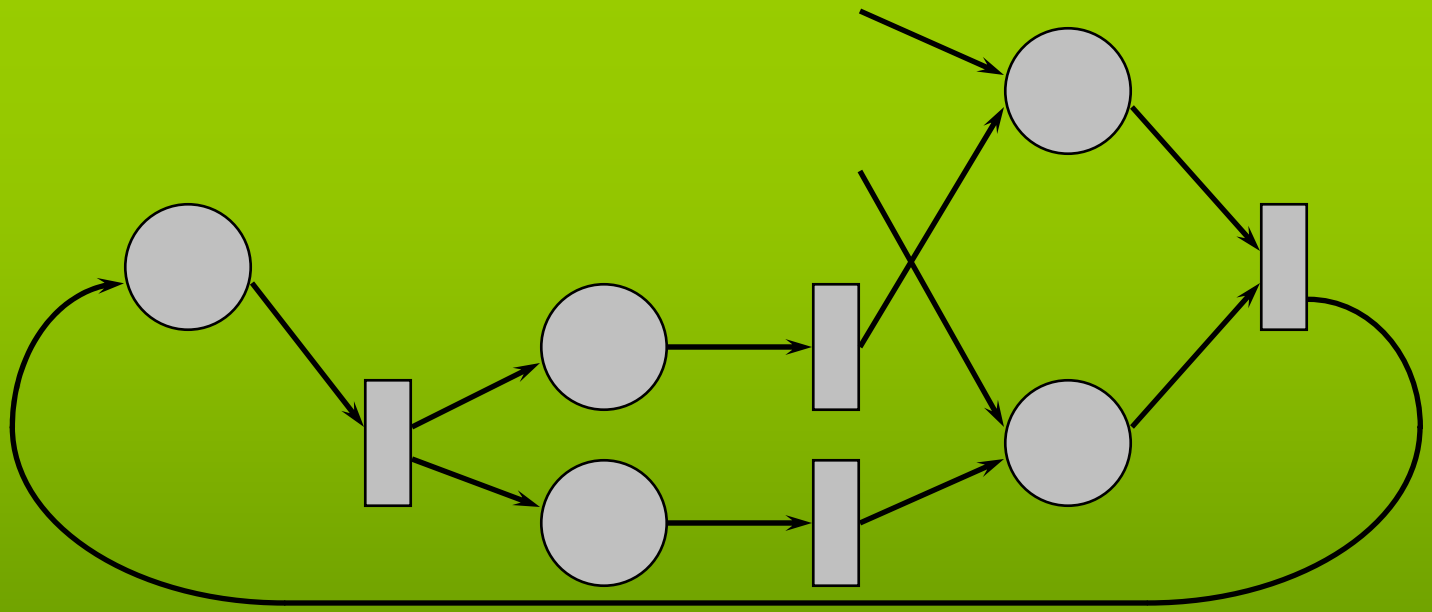
MG reduction and cover

- **Choose one successor for each conflicting place:**



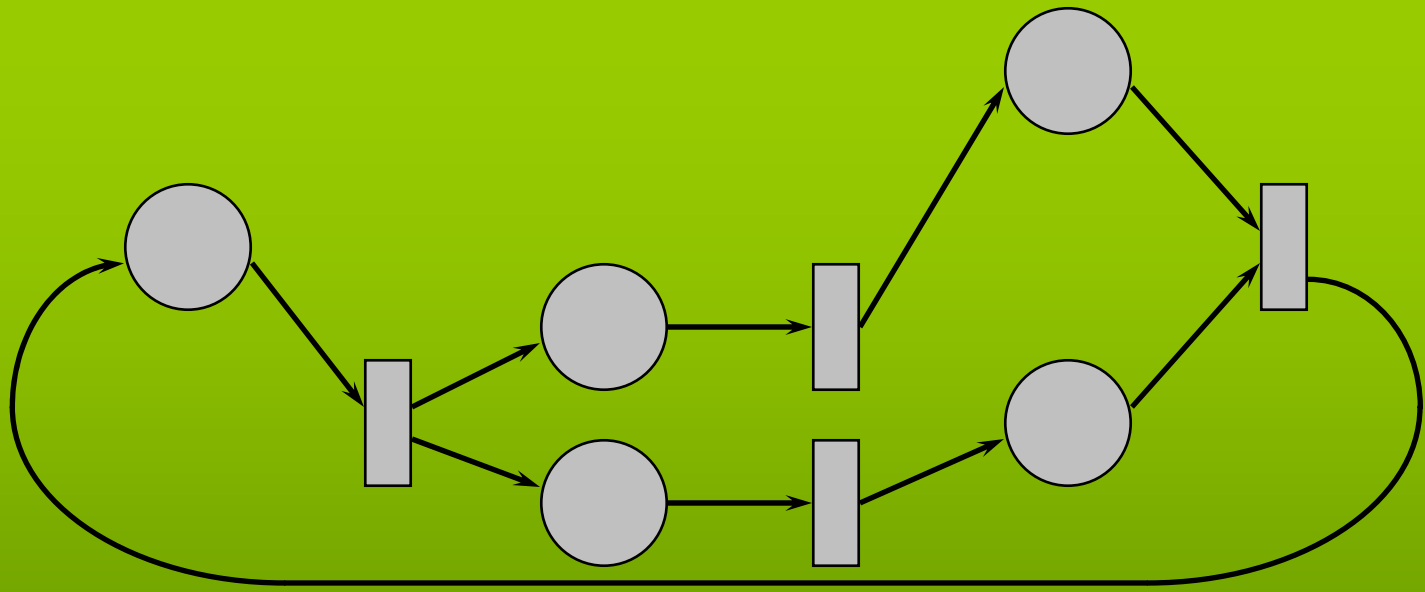
MG reduction and cover

- **Choose one successor for each conflicting place:**



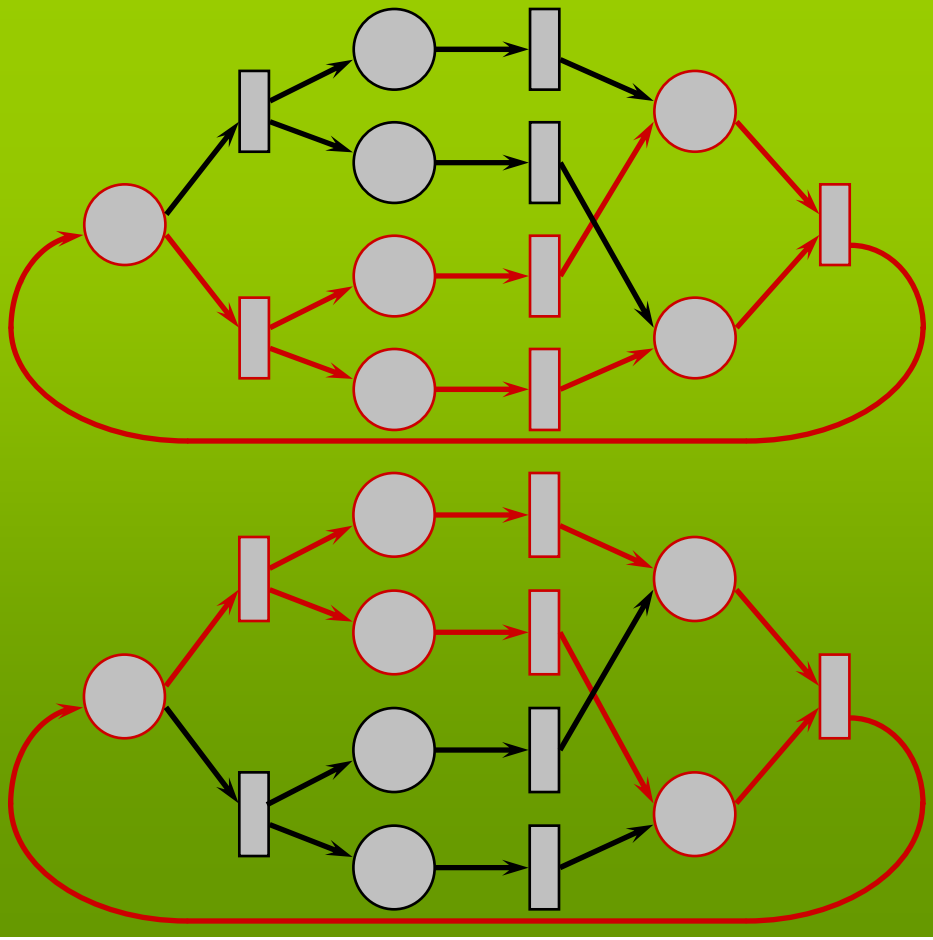
MG reduction and cover

- **Choose one successor for each conflicting place:**



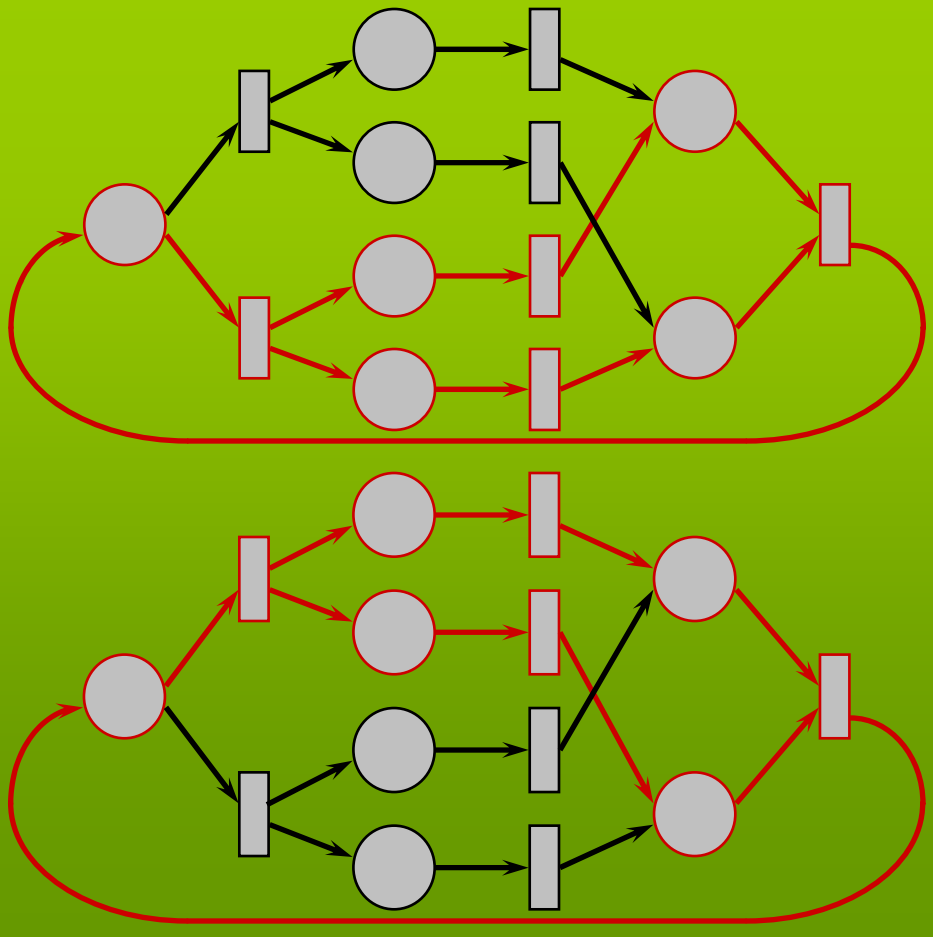
MG reductions

- **The set of all reductions yields a cover of MG components (T-invariants)**



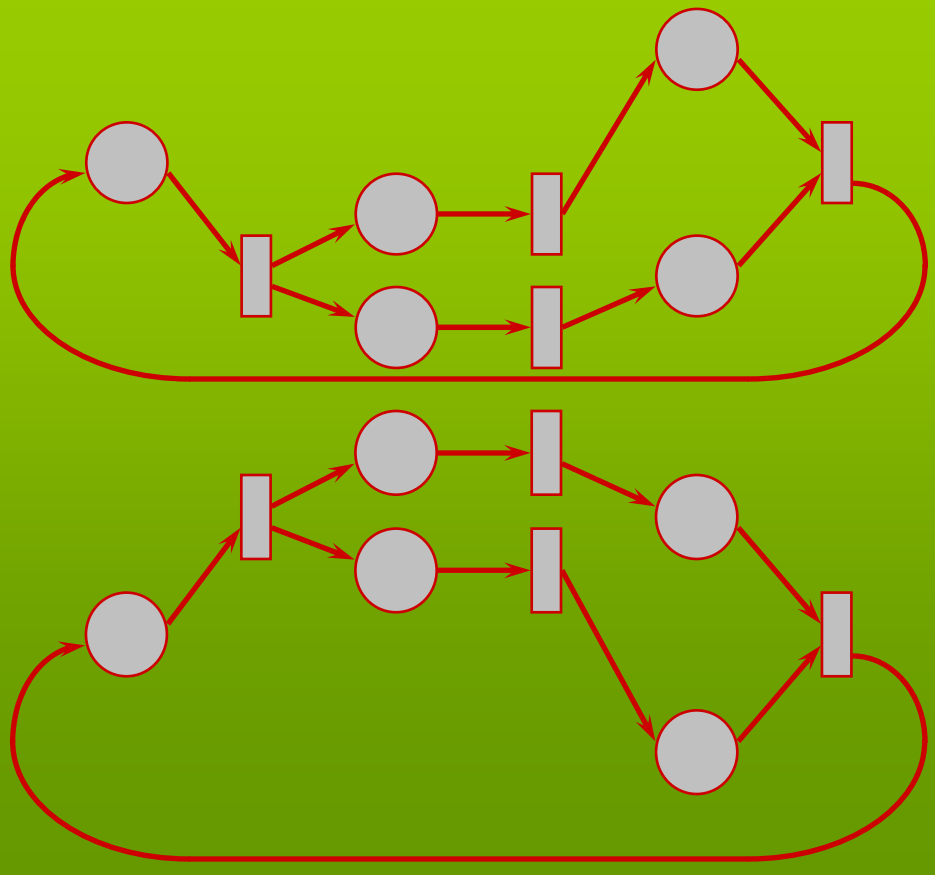
MG reductions

- **The set of all reductions yields a cover of MG components (T-invariants)**



MG reductions

- **The set of all reductions yields a cover of MG components (T-invariants)**



Hack's theorem ('72)

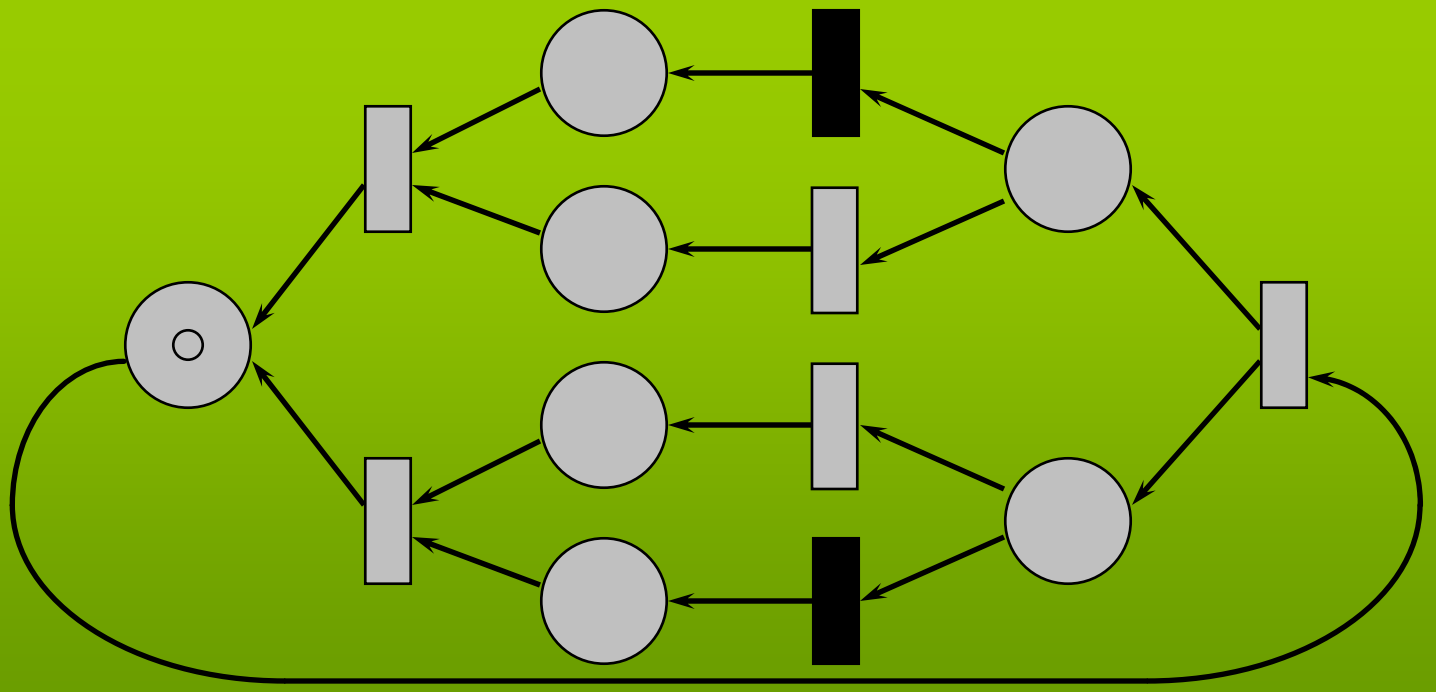
- **Let N be a Free-Choice PN:**
  - **N has a live and safe initial marking (well-formed)**

  if and only if

  - **every MG reduction is strongly connected and not empty, and the set of all reductions covers the net**
  - **every SM reduction is strongly connected and not empty, and the set of all reductions covers the net**

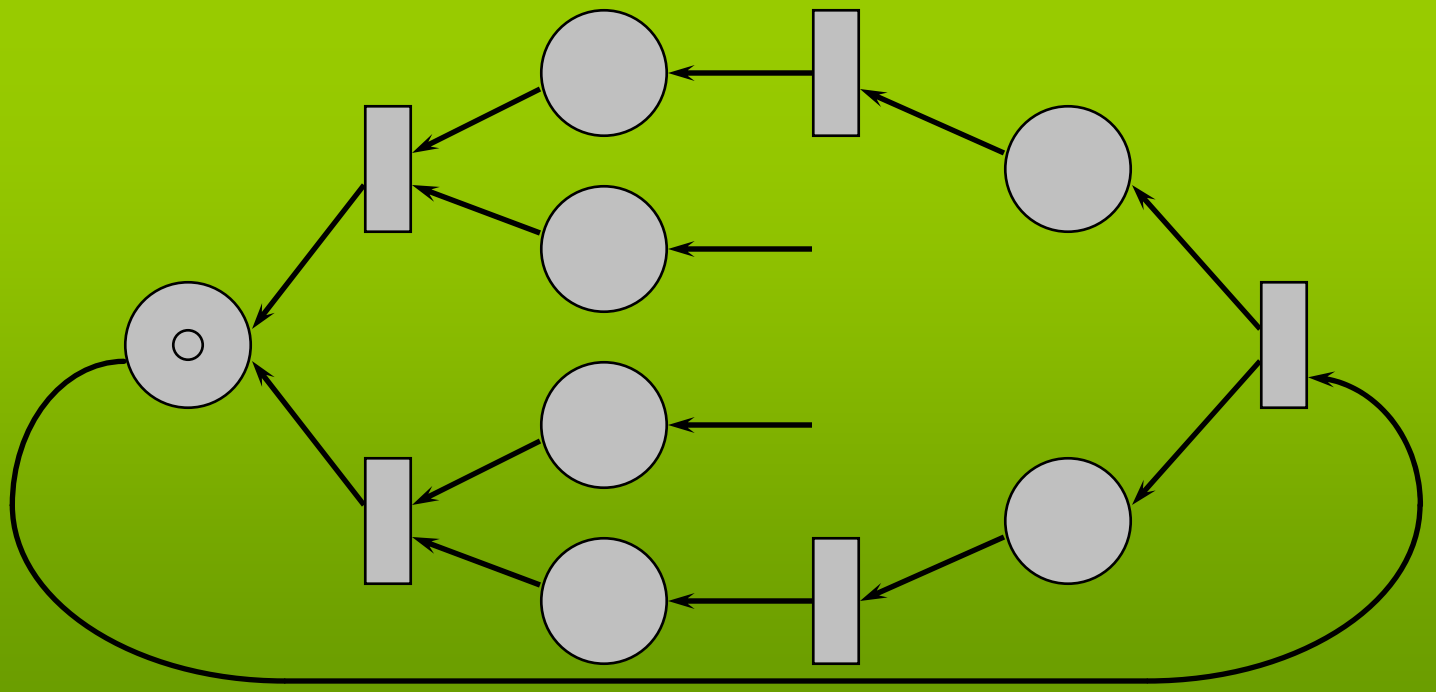
Hack's theorem

- **Example of non-live (but safe) FCN**



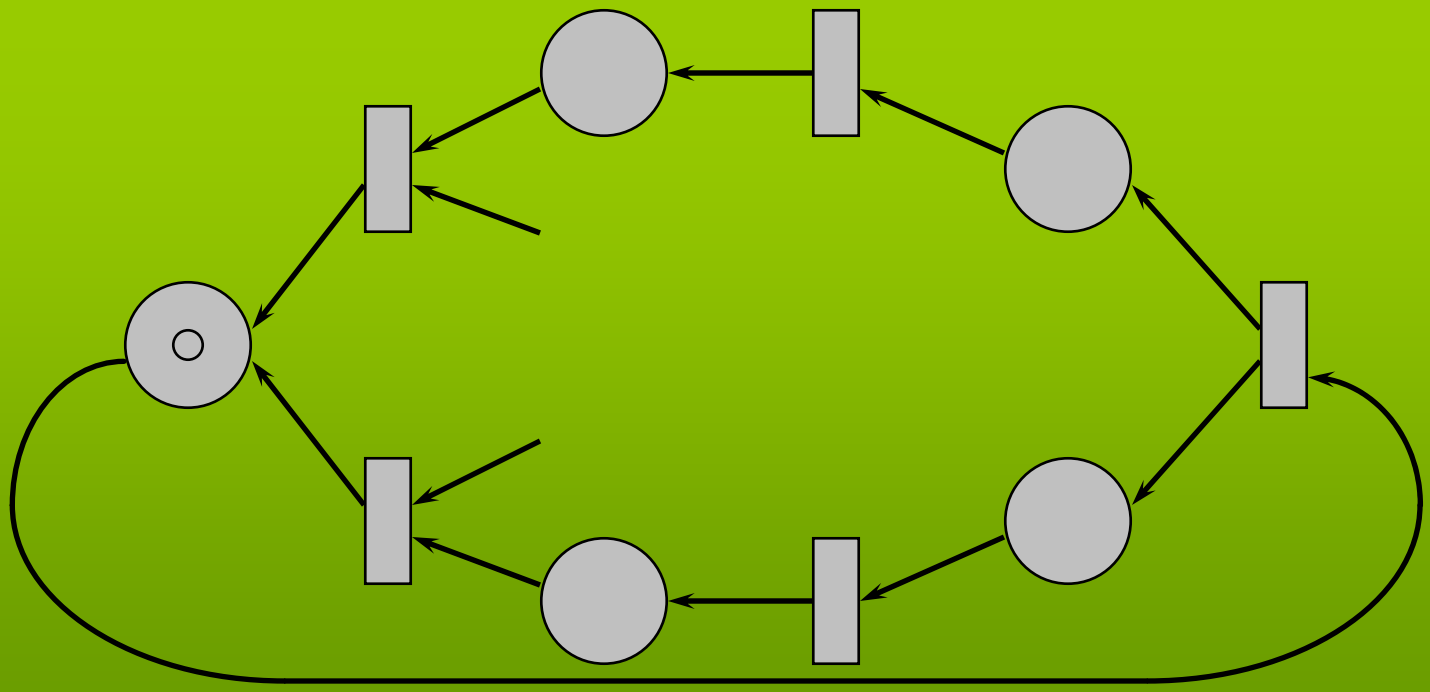
Hack's theorem

- **Example of non-live (but safe) FCN**



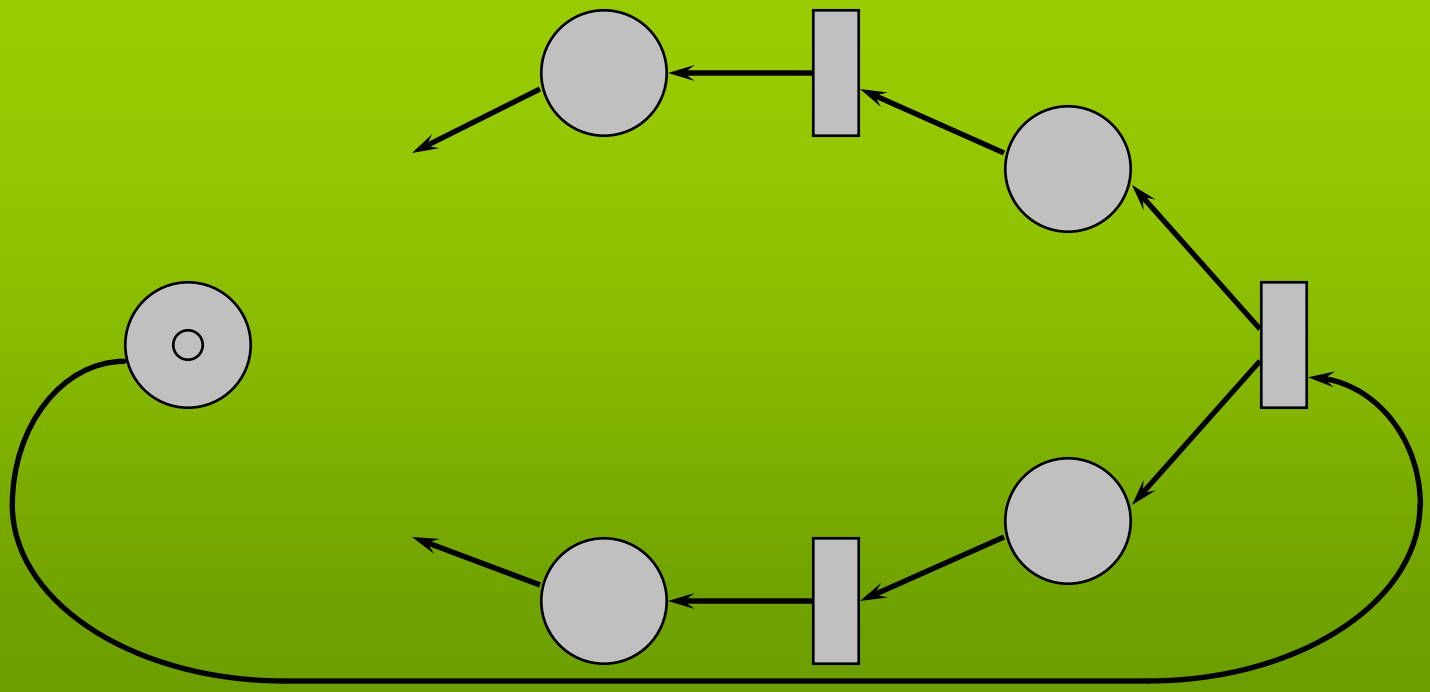
Hack's theorem

- **Example of non-live (but safe) FCN**



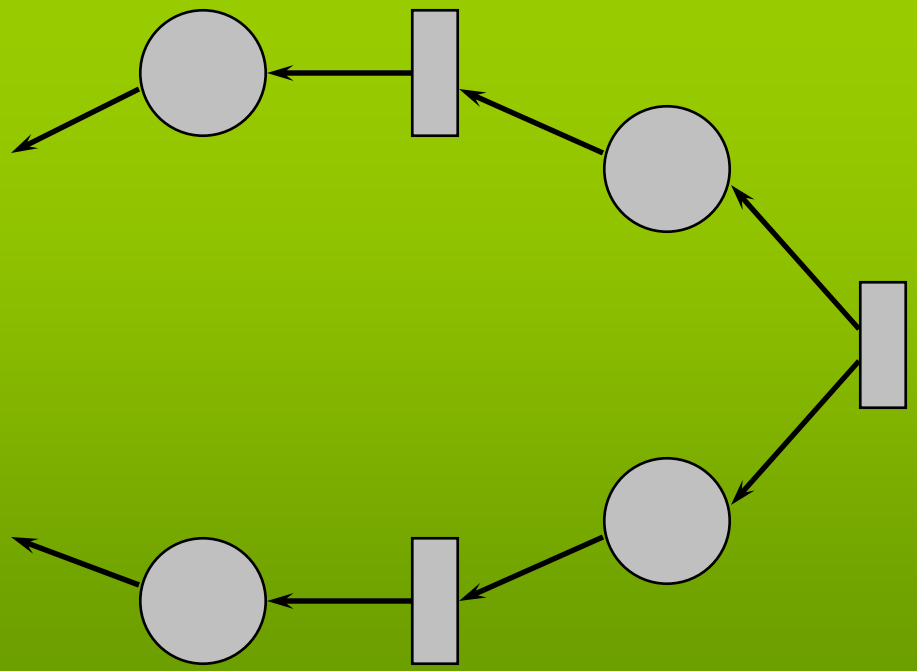
Hack's theorem

- **Example of non-live (but safe) FCN**



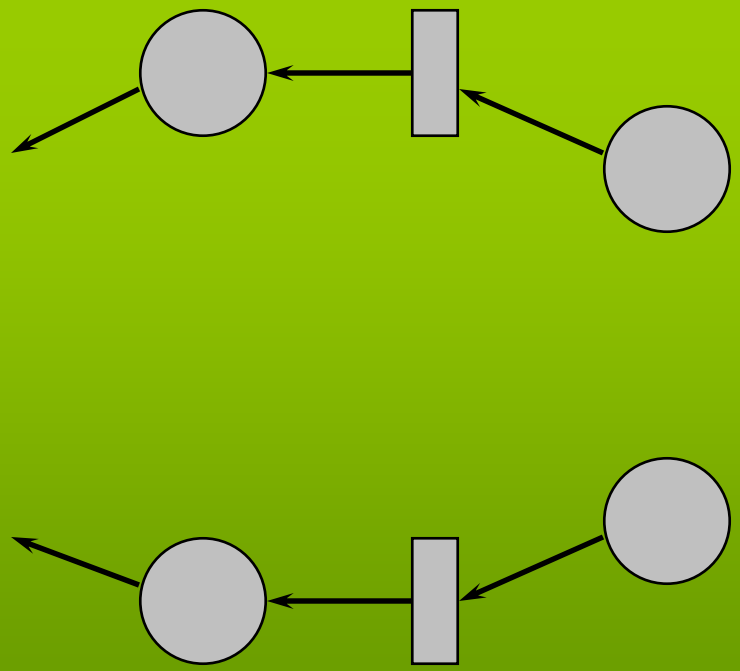
Hack's theorem

- **Example of non-live (but safe) FCN**



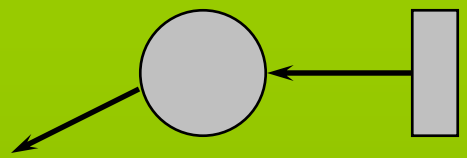
Hack's theorem

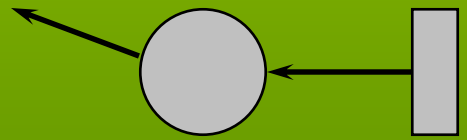
- **Example of non-live (but safe) FCN**



Hack's theorem

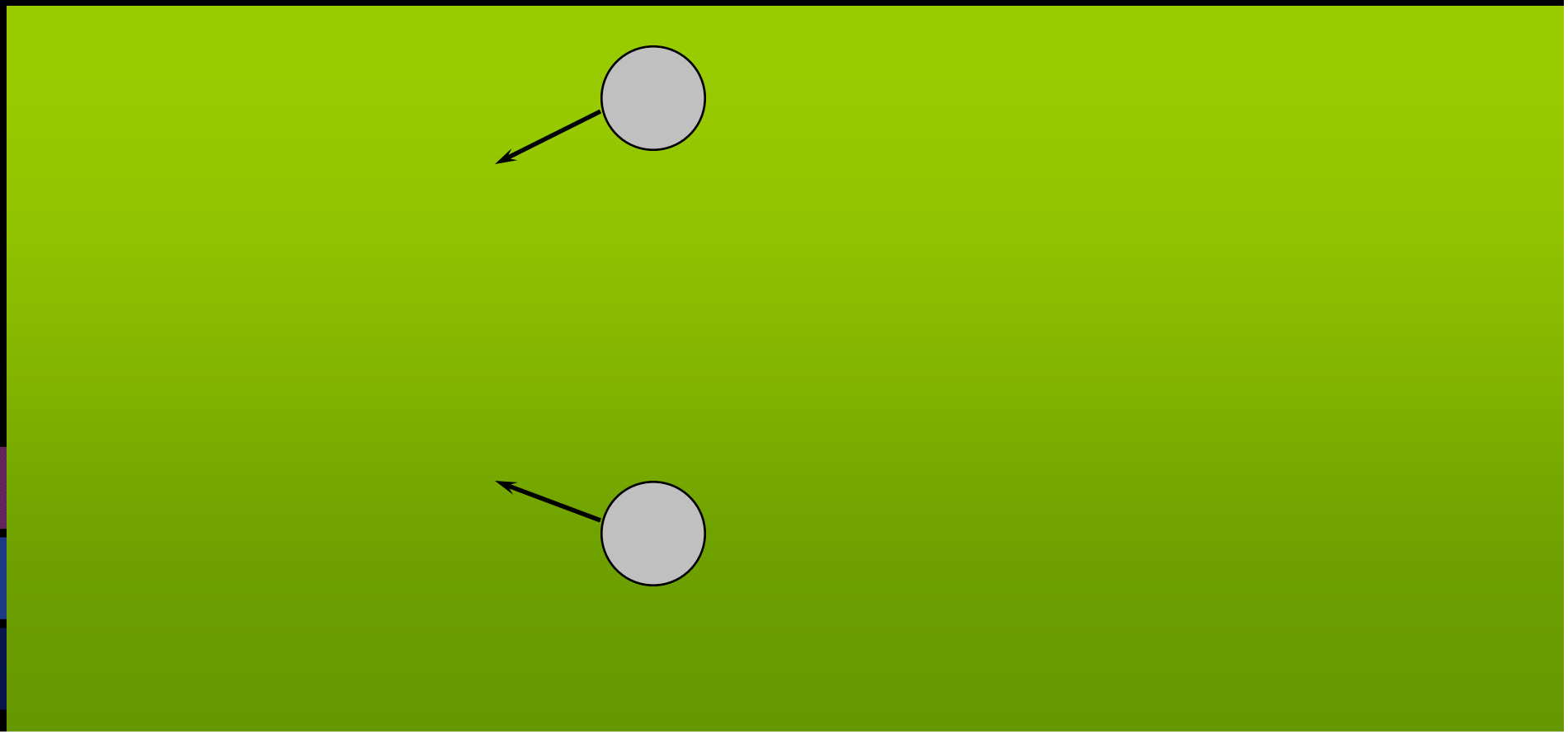
- **Example of non-live (but safe) FCN**





Hack's theorem

- **Example of non-live (but safe) FCN**

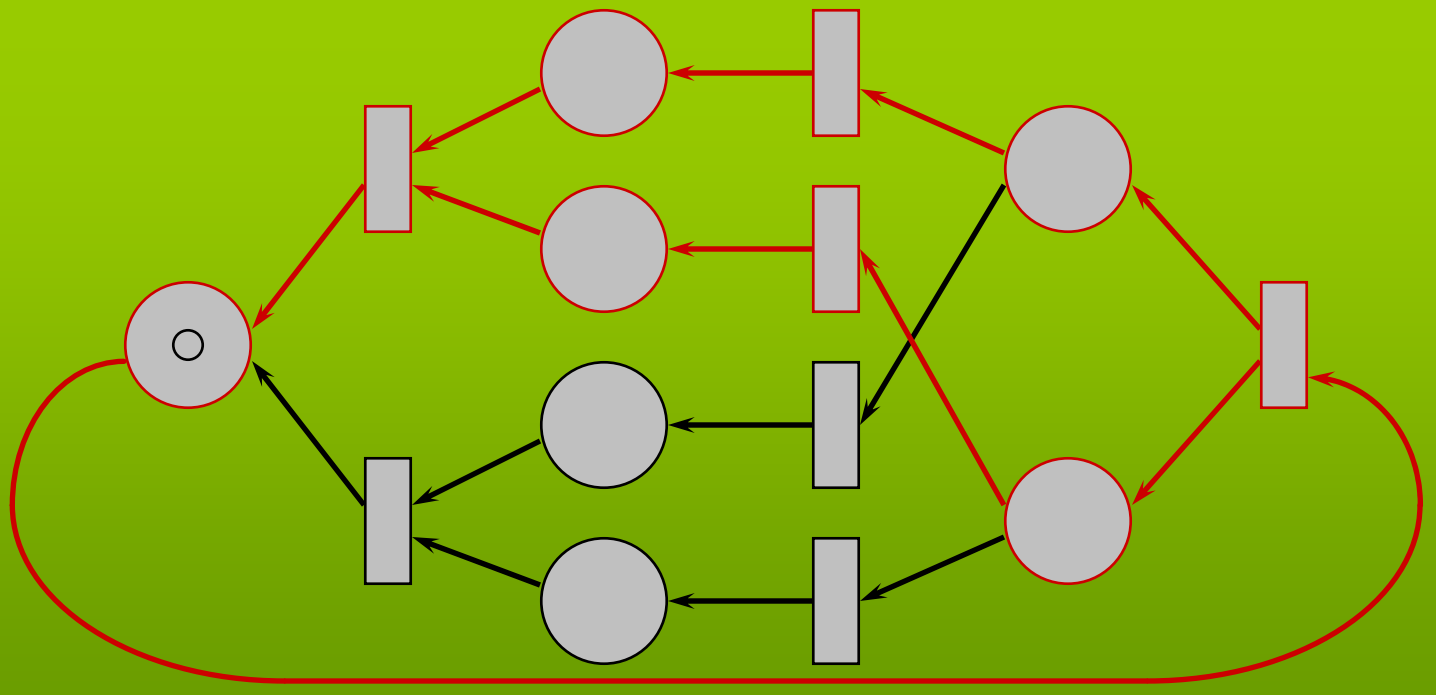


Hack's theorem

- **Example of non-live (but safe) FCN**

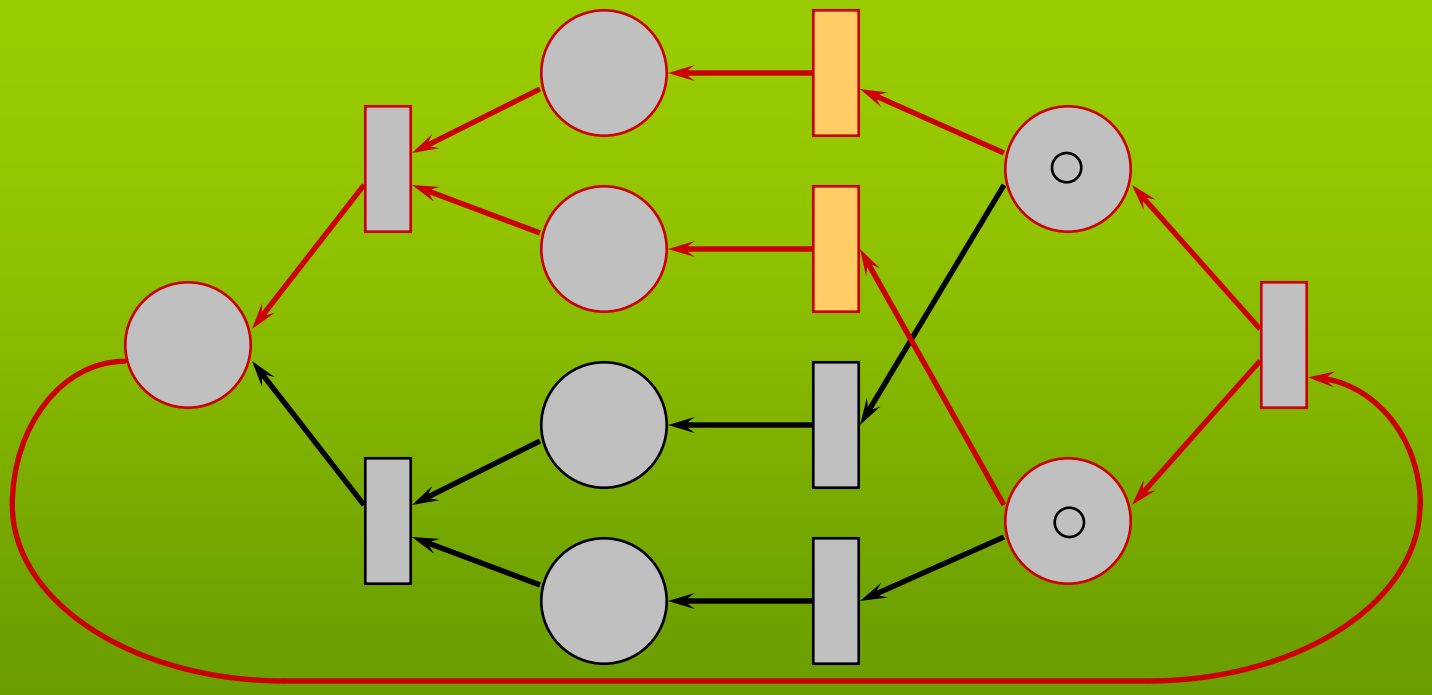
Hack's theorem

- **Example of non-live (but safe) FCN**



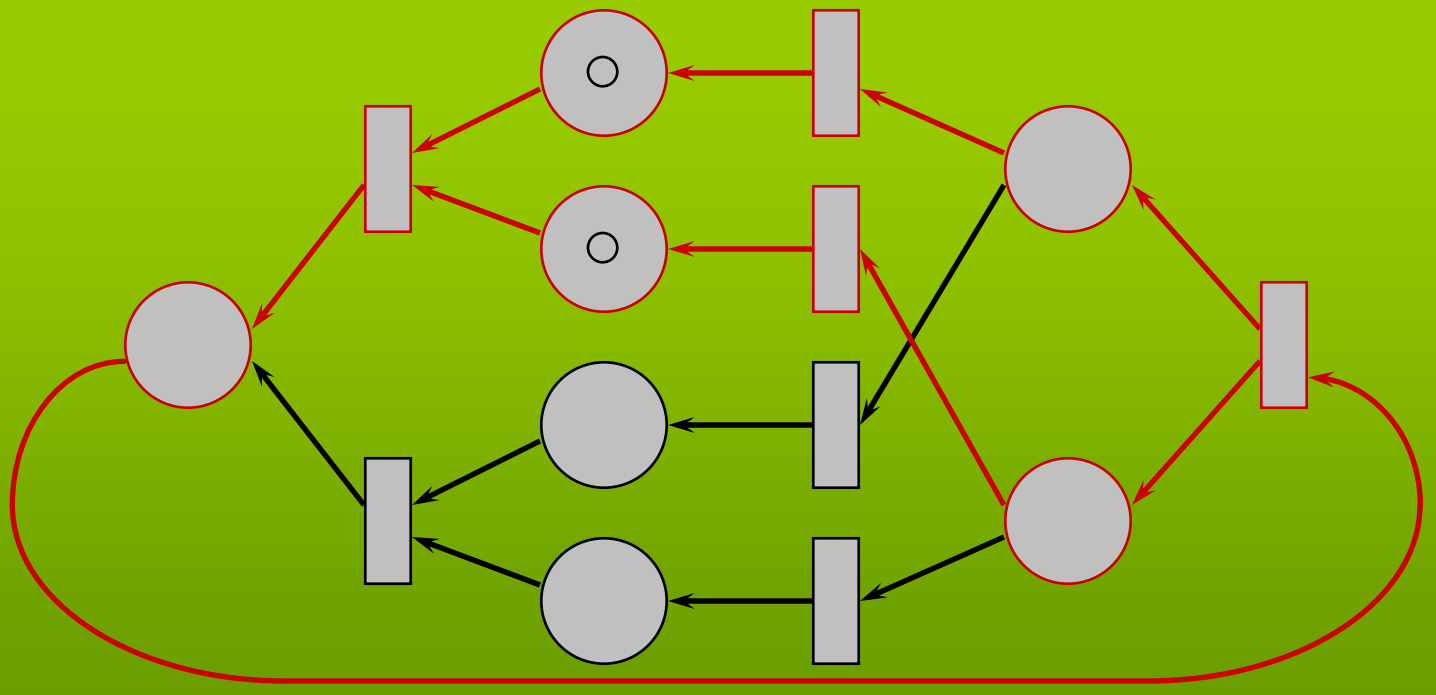
Hack's theorem

- **Example of non-live (but safe) FCN**



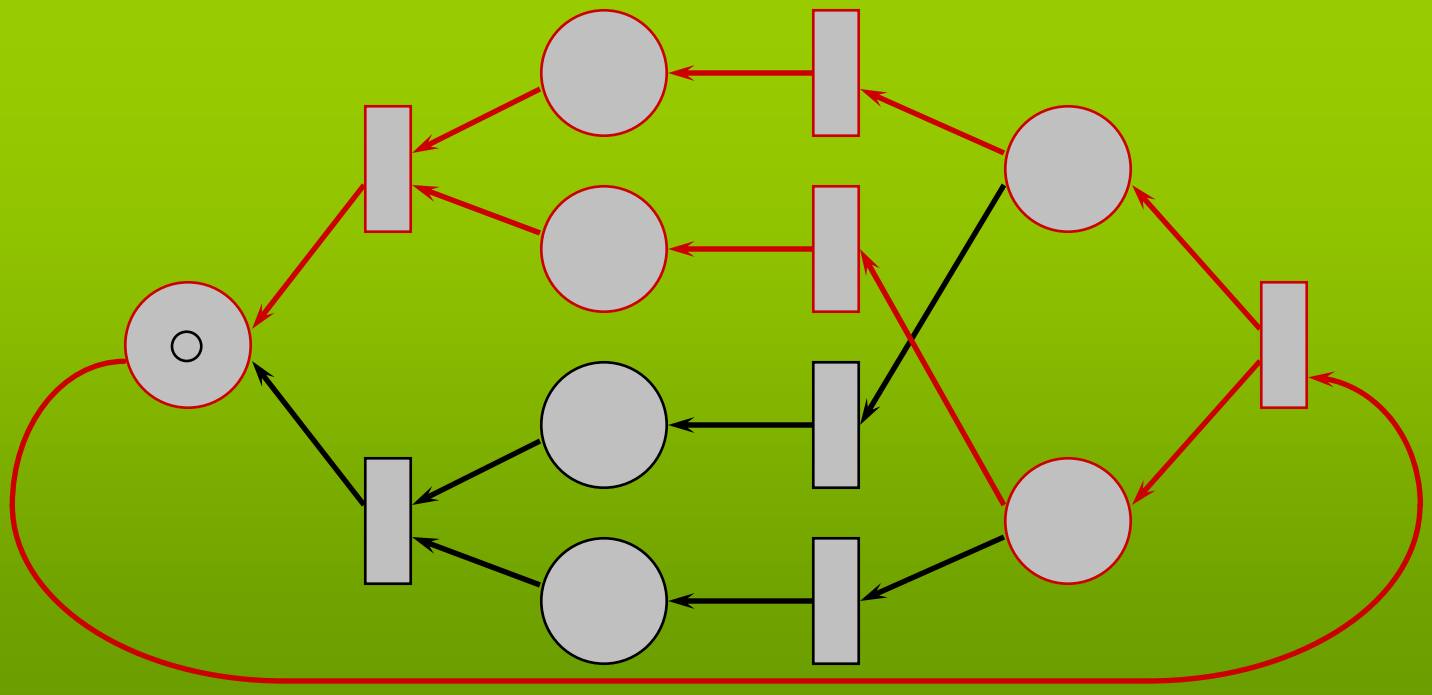
Hack's theorem

- **Example of non-live (but safe) FCN**



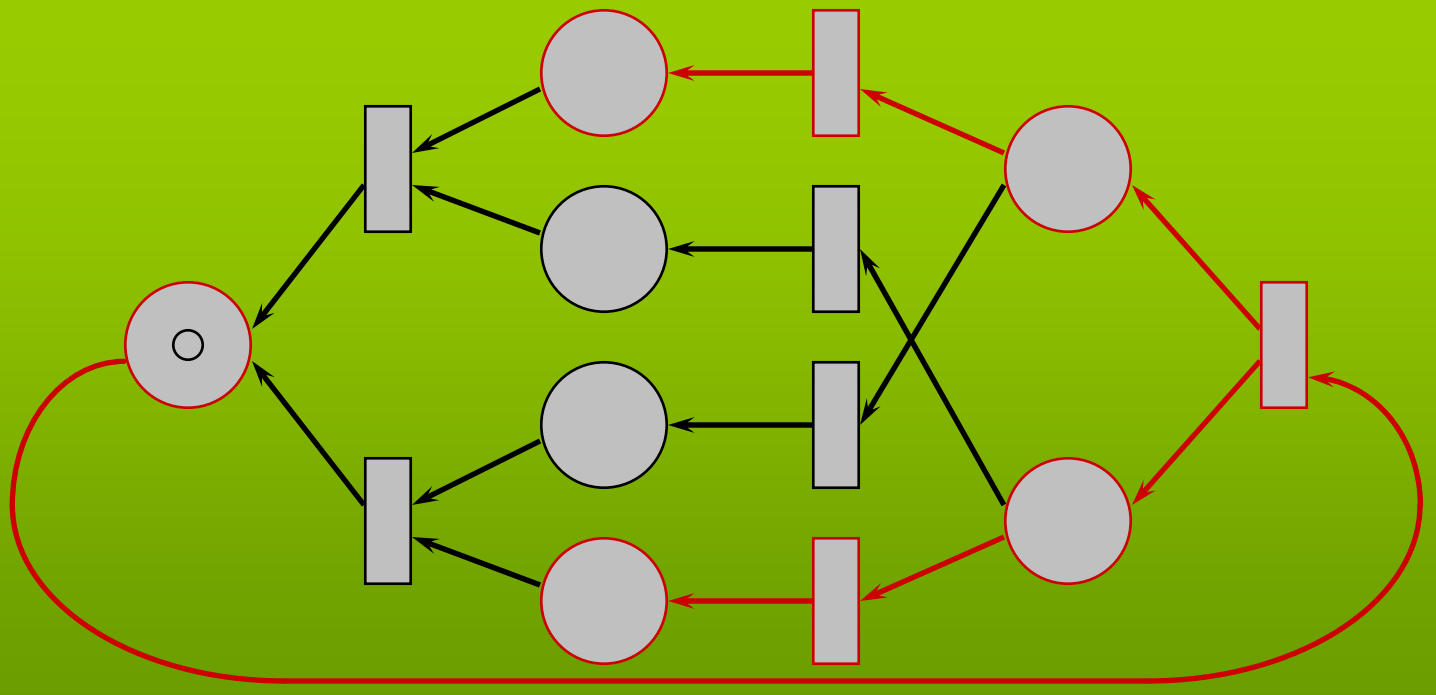
Hack's theorem

- **Example of non-live (but safe) FCN**



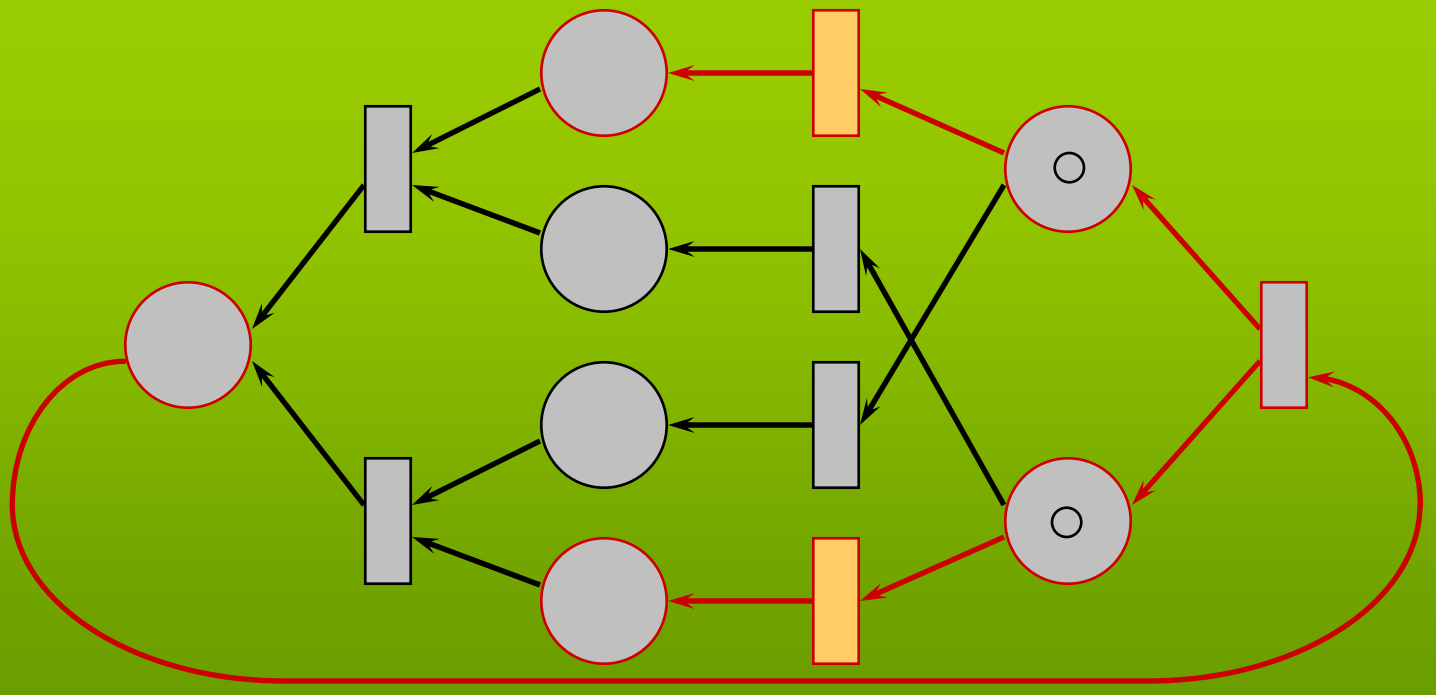
Hack's theorem

- **Example of non-live (but safe) FCN**



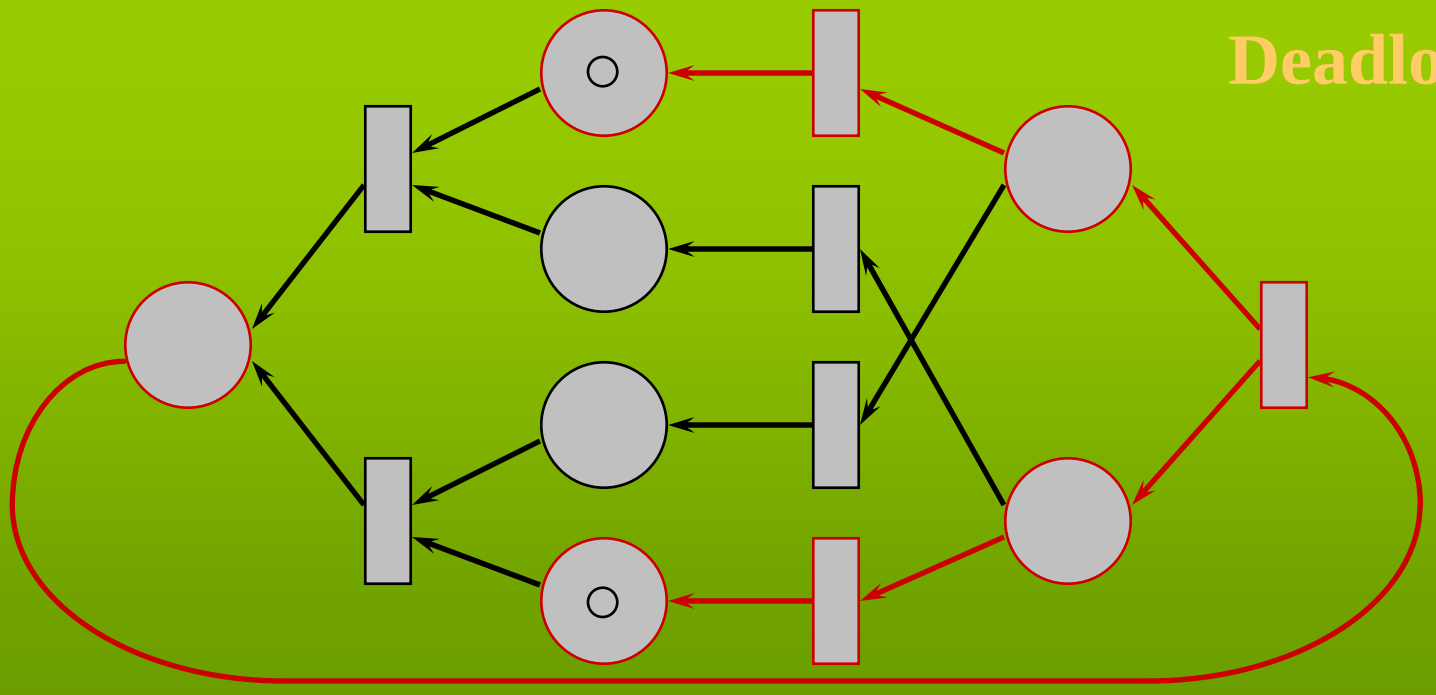
Hack's theorem

- **Example of non-live (but safe) FCN**



Hack's theorem

- **Example of non-live (but safe) FCN**



Summary of LSFC nets

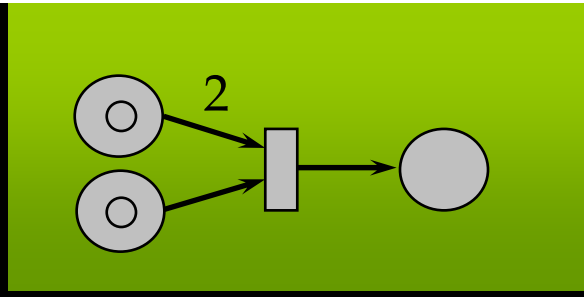
- **Largest class for which structural theory really helps**
- **Structural component analysis may be expensive**

  (exponential number of MG and SM components in the worst case)
- **But…**
  - **number of MG components is generally small**
  - **FC restriction simplifies characterization of behavior**

Petri Net extensions

- **Add interpretation to tokens and transitions**
  - **Colored nets (tokens have value)**
- **Add time**
  - **Time/timed Petri Nets (deterministic delay)**
    - **type (duration, delay)**
    - **where (place, transition)**
  - **Stochastic PNs (probabilistic delay)**
  - **Generalized Stochastic PNs (timed and immediate transitions)**
- **Add hierarchy**
  - **Place Charts Nets**

PNs Summary



- **PN Graph: places (buffers), transitions (actions), tokens (data)**
- **Firing rule: transition enabled if there are enough tokens in each input place**
- **Properties**
  - **Structural (consistency, structural boundedness…)**
  - **Behavioral (reachability, boundedness, liveness…)**
- **Analysis techniques**
  - **Structural (only CN or CS): State equations, Invariants**
  - **Behavioral: coverability tree**
- **Reachability**
- **Subclasses: Marked Graphs, State Machines, Free-Choice PNs**

References

- T. Murata **Petri Nets: Properties, Analysis and Applications**
- http://www.daimi.au.dk/PetriNets/