EE219B

Logic Synthesis

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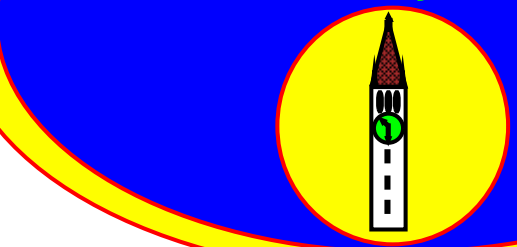


SPRING, 1998

Introduction

Motivation

- commercial success - used almost everywhere VLSI is done
- but more general - generally - discrete functions of discrete valued variables
- body of useful and general techniques - can be applied to other areas
- foundation for:
  - combinational and sequential synthesis
  - testing
  - timing and false paths
  - formal verification
  - asynchronous synthesis
  - automata theory
  - optimal clocking schemes
  - hazard analysis
  - power estimation
  - general combinatorics.

Outline of Class

The entire class notes are available at
http://www-cad.eecs.berkeley.edu/HomePages/brayton/219b/219b.html

1. Introduction
   - logic functions and their representation
   - unate recursive paradigm, Quine McCluskey

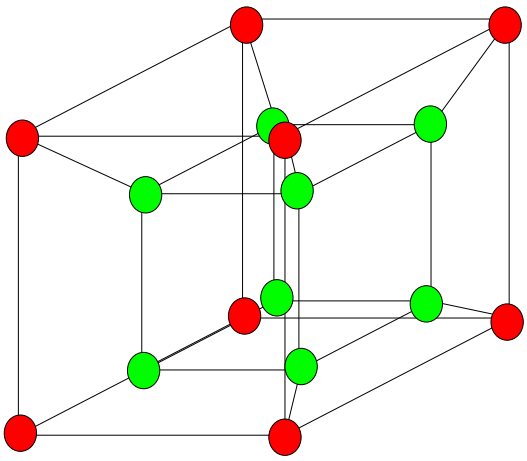
2. Two Level Minimization
   - ESPRESSO two level minimization
   - multivalued minimization
   - BDD's

3. Multi-Level Logic Synthesis
   - introduction (Boolean networks, factored forms)
   - division
   - simplification
   - full simplify
   - testing
   - technology mapping
   - advanced technology mapping
   - delay analysis
   - timing optimization

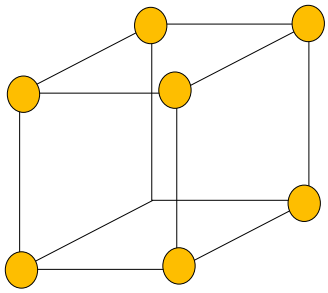
4. Sequential Logic Synthesis
   - introduction (FSM networks)
   - node minimization
   - state minimization
   - retiming and resynthesis

The Boolean n-cube

B^n

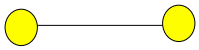


B^4

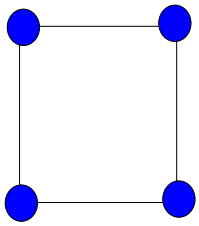


B^3

B^0



B^1



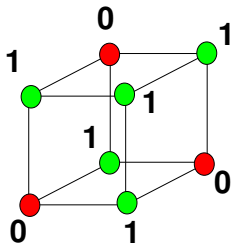
B^2

- $\mathcal{B} = \{0, 1\}$
- $\mathcal{B}^2 = \{0, 1\} \times \{0, 1\} = \{00, 01, 10, 11\}$

Boolean Functions

$f(x) : \mathcal{B}^n \rightarrow \mathcal{B}$
$\mathcal{B} = \{0, 1\}$, $x = (x_1, x_2, \ldots, x_n)$

- each **vertex** of $\mathcal{B}^n$ is mapped to 0 or 1.



- the **onset** of f is $\{x \mid f(x) = 1\} = f^1 = f^{-1}(1)$
- the **offset** of f is $\{x \mid f(x) = 0\} = f^0 = f^{-1}(0)$
- if $f^1 = \mathcal{B}^n$, f is the **tautology**, i.e. $f \equiv 1$
- if $f^0 = \mathcal{B}^n$, $(f^1 = \phi)$ f is **not satisfiable**
- if $f(x) = g(x)$ for all $x \in \mathcal{B}^n$, then f and g are **equivalent**
- $x_1, x_2, \ldots$ are **variables**
- $x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots$ are **literals**

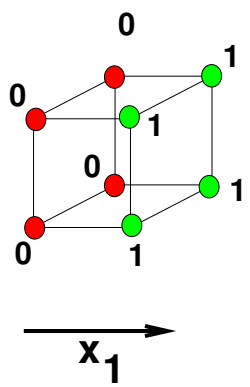
Literals

A literal is a variable or its negation

$$y, \overline{y}$$

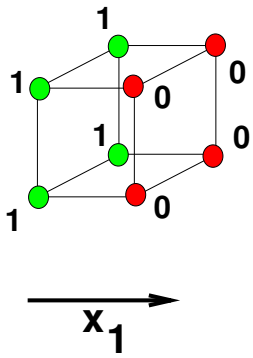
It represents a **logic function**

Literal x_1 represents the logic function f, where $f^1 = \{x \mid x_1 = 1\}$



f = x_1

Literal $\overline{x}_1$ represents logic function g where $g^1 = \{x \mid x_1 = 0\}$



f = $\overline{x}_1$

Boolean Formulas

Boolean functions can be **represented** by **formulas** defined as catenations of

- parentheses - (,)
- literals - $x, y, z, \overline{x}, \overline{y}, \overline{z}$
- Boolean operators - $+$ (OR), $\times$ (AND)
- complementation - e.g. $\overline{x + y}$

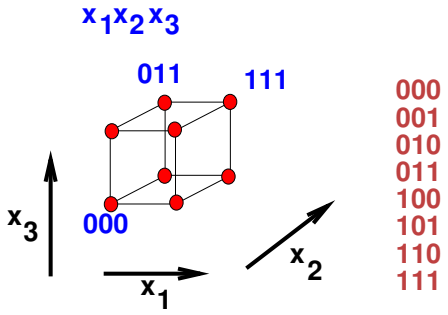
Examples:

$$\begin{aligned} f &= x_1 \times \overline{x}_2 + \overline{x}_1 \times x_2 \\ &= (x_1 + x_2) \times (\overline{x}_1 + \overline{x}_2) \\ h &= a + b \times c \\ &= \overline{\overline{a} \times (\overline{b} + \overline{c})} \end{aligned}$$

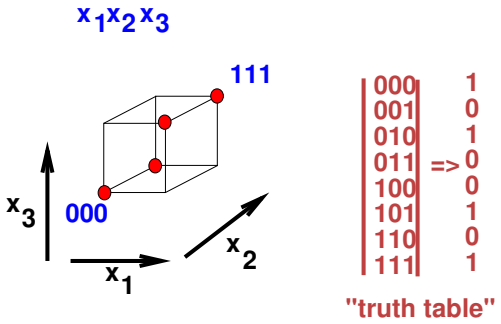
We will usually replace $\times$ by catenation, e.g. $a \times b \rightarrow ab$.

Logic Functions - $f(x) : B^n \rightarrow B$

- There are 2^n vertices in input space B^n



- There are 2^{2^n} distinct logic **functions**. Each subset of vertices is a distinct logic function: $f^1 \subseteq B^n$



- There are ∞ number of logic **formulas**

$$\begin{aligned} f &= x + y \\ &= x\overline{y} + xy + \overline{x}y \\ &= x\overline{x} + x\overline{y} + y \\ &= (x + y)(x + \overline{y}) + \overline{x}y \end{aligned}$$

SYNTHESIS = Find the "best" formula (or "representation")

Boolean Operations - AND, OR, COMPLEMENT

$$f \quad : \quad B^n \rightarrow B$$
$$g \quad : \quad B^n \rightarrow B$$

- AND - $f \times g = h$ such that
  $h^1 = \{x \mid f(x) = 1 \text{ and } g(x) = 1\}$

- OR - $f + g = h$ such that
  $h^1 = \{x \mid f(x) = 1 \text{ or } g(x) = 1\}$

- COMPLEMENT - $\overline{f} = h$ such that
  $h^1 = \{x \mid f(x) = 0\}$

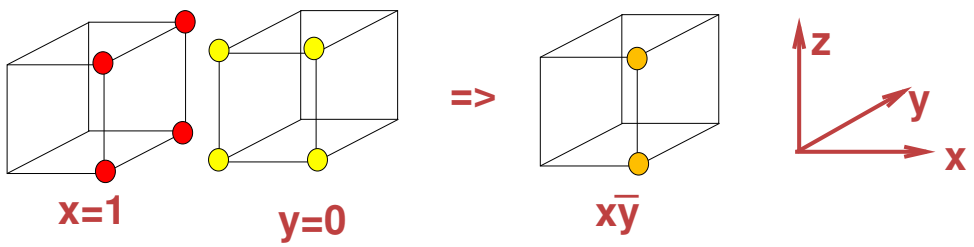
Cubes

- The AND of a set of literal functions ("conjunction" of literals) is a **cube**

$$C = x\overline{y}$$

is a cube

$$C = (x = 1)(y = 0)$$



- if $C^1 \subseteq f^1$, C a cube, then C is an **implicant** of f
- if $C^1 \subseteq B^n$ and C has k literals, then $|C^1|$ has 2^{n-k} vertices

Example 1 $C = x\overline{y} \subseteq B^3$
$k = 2$
$n = 3$

$$C^1 = \{100, 101\}$$

$|C^1| = 2 = 2^{3-2}$

- if $k = n$, the cube is a **min-term**

Representation of Boolean Functions

The **truth table** of a function $f : B^n \to B$ is a tabulation of its value at each of the 2^n vertices of B^n.

For

$$f = \overline{a}\overline{b}\overline{c}d + \overline{a}\overline{b}cd + \overline{a}b\overline{c}d + a\overline{b}\overline{c}d + a\overline{b}cd + ab\overline{c}d + abc\overline{d} + abcd$$

the truth table is

	abcd	f		abcd	f
0	0000	0	8	1000	0
1	0001	1	9	1001	1
2	0010	0	10	1010	0
3	0011	1	11	1011	1
4	0100	0	12	1100	0
5	0101	1	13	1101	1
6	0110	0	14	1110	1
7	0111	0	15	1111	1

This is intractable for large n (but canonical).

Canonical means that if two functions are the same, then the canonical representations of each are isomorphic.

Sum-of-Products Representation - SOP

- A function can be represented by a sum of cubes (products):

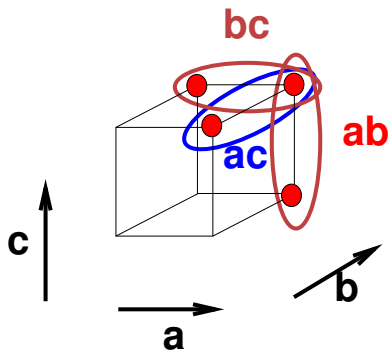
$$f = ab + ac + bc$$

  Since each cube is a product of literals, this is a "sum-of-products' representation

- A SOP can be thought of as a set of cubes

$$f = \{ab, ac, bc\} = \mathcal{C}$$

- A set of cubes that represents f is called a **cover** of f. $\{ab, ac, bc\}$ is a cover of $f = ab + ac + bc$.



● = **onset vertex**

Note that each onset vertex is "covered" by at least one of the cubes, and covers no offset vertex.

- Covers (SOP's) can efficiently represent many logic functions (i.e. for many, there exist small covers).

- Two-level minimization seeks the minimum size cover (least # of cubes).

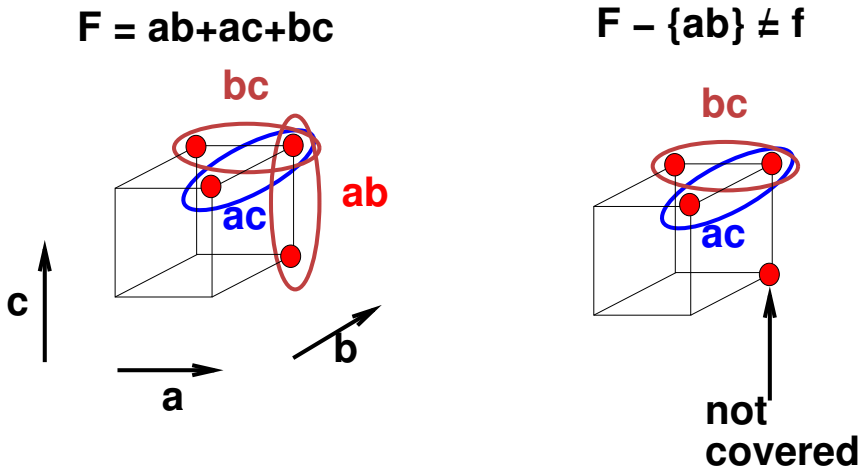
Irredundant

Let $F = \{c_1, c_2, \ldots, c_k\}$ be a cover for f

$$f = \sum_{i=1}^{k} c_i$$

A cube $c_i \in F$ is **irredundant** if $F - \{c_i\} \neq f$

Example 2



Prime

An implicant (cube) of f is **prime** if

$$f + c_i' \neq f$$

where c_i' is c_i with any literal of c_i deleted.

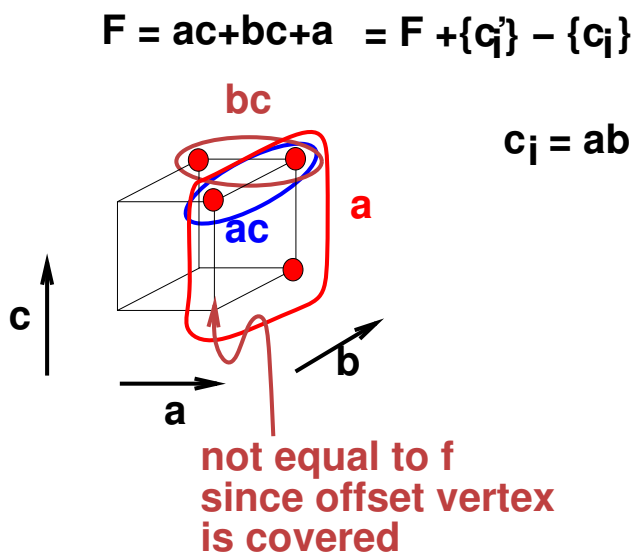
Example 3

$f = ab + ac + bc$

$c_i = ab$

$c_i' = a$ *(literal b deleted)*

$F \cup \{c_i'\} - \{c_i\} = a + ac + bc$

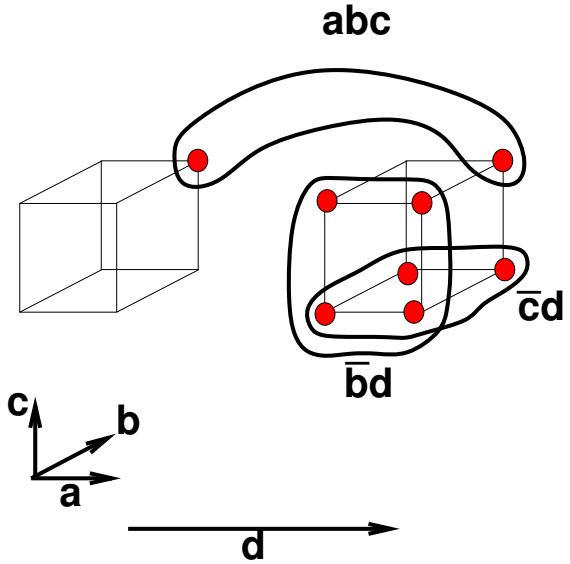


Prime and Irredundant Covers

Definition 1 *A* **cover** *is prime* **irredundant** *if all its cubes are prime (irredundant).*

Definition 2 *A prime of* f *is* **essential** *(essential prime) if there is a min-term (essential vertex) in that prime but in no other prime.*

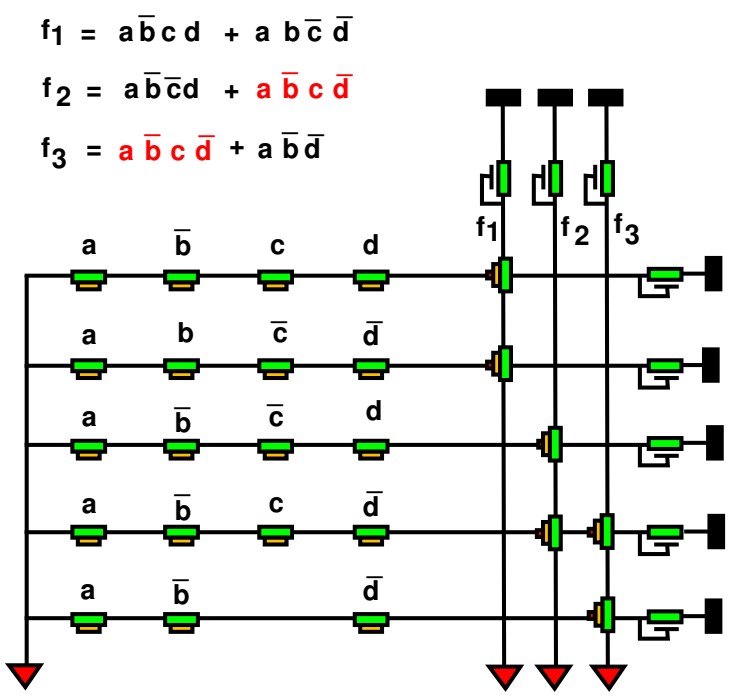
Example 4 $F = abc + \bar{b}d + \bar{c}d$
is prime and irredundant
abc *is essential since* $abc\bar{d} \in abc$ *but not in* $\bar{b}d$ *or* $\bar{c}d$ *or* ad



Why is $abcd$ *not an essential point of* abc*?*
What is an essential vertex of abc*?*
What other cube is essential?
What cube is not essential?

PLA's - Multiple Output Boolean Functions

A PLA is a function $f : B^n \rightarrow B^m$ represented in SOP form:



- Each distinct cube appears just once in the **AND-plane**, and can be shared by (multiple) outputs in the **OR-plane**, e.g. cube $a\overline{b}c\overline{d}$.

- Extensions of single output minimization theory are straightforward.

- Multilevel logic can be viewed mathematically as a connection of single output functions.

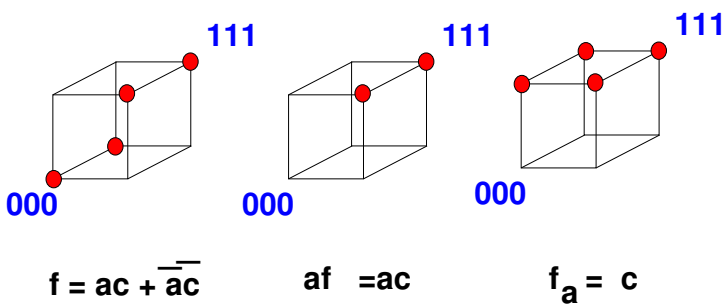
Shannon (Boole) Cofactors

Let $f : B^n \rightarrow B$ be a Boolean function, and $x = (x_1, \ldots, x_n)$ the variables in the support of f. The **cofactor** f_a of f by a literal $a = x_i$ or $a = \overline{x}_i$ is

$$f_{x_i}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$
$$f_{\overline{x}_i}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

Notes:

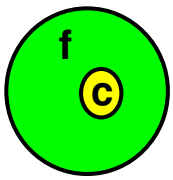
- The cofactor f_C of f by a cube C is just f with the fixed values indicated by the literals of C, e.g. if $C = x_i\overline{x}_j$, then $x_i = 1$ and $x_j = 0$.
- If $C = x_1\overline{x}_4x_6$, f_C is just the function f restricted to the subspace where $x_1 = x_6 = 1$ and $x_4 = 0$.
- As a function, f_C does not depend on x_1, x_4, or x_6. (*However, we still consider f_C as a function of all n variables, it just happens to be independent of x_1, x_4, x_6*)
- $x_1 f \neq f_{x_1}$



Fundamental Theorem

Theorem 1 *Let c be a cube and f a function. Then* $c \subseteq f \iff f_c \equiv 1$.

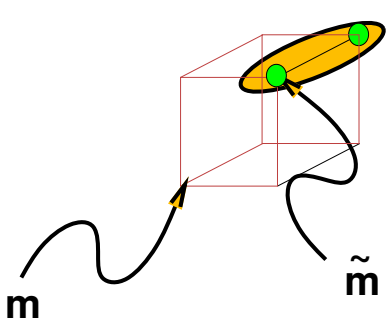
Proof. We use: $xf_x = xf$ and f_x is independent of x.
if: Suppose $f_c \equiv 1$. Then $f_c c = c$ and $c = cf$. Thus $c \subseteq f$.



only if: Assume $c \subseteq f$. Then $c \subseteq fc = f_c c$. But f_c is independent of literals $l \in c$. If $f_c \neq 1$, then $\exists m \in B^n$, $f_c(m) = 0$. Let
$\widetilde{m}_i = m_i$, if $x_i \notin c$ and $\overline{x}_i \notin c$
$\widetilde{m}_i = m_i$, if $m_i = 0$, $\overline{x}_i \in c$ or $m_i = 1, x_i \in c$
$\widetilde{m}_i = \overline{m}_i$, otherwise
i.e. we make the literals of $\widetilde{m}$ agree with c. But then

$$f_c(\widetilde{m}) = f_c(m) = 0$$

and $\widetilde{m} \in c$, Hence $c(\widetilde{m}) = 1$ and $f_c(\widetilde{m})c(\widetilde{m}) = 0$ contradicting $c \subseteq f_c c$. □



c = xz

m=0 0 0

m̃=1 0 1

Cofactor of Covers

Definition 3 *The* **cofactor of a cover** F *is the sum of the cofactors of each of the cubes of* F.

Note: If $F = \{c_1, c_2, \ldots, c_k\}$ is a cover of f, then $F_c = \{(c_1)_c, (c_2)_c, \ldots, (c_k)_c\}$ is a cover of f_c.

Suppose $F(x) : B^n \rightarrow B$ is a cover, i.e.

$$F(x) = \sum_i C_i = \sum_i \prod_j l_j^i = \{C_i\}$$

Then for $1 \leq j \leq n$,

$$F(x)_{x_j} = \sum_i (C_i)_{x_j}$$

Definition 4 *The* **cofactor** C_{x_j} *of a cube* C *with respect to a literal* x_j *is*

- C *if* x_j *does not appear in* C
- $C - \{x_j\}$ *if* x_j *appears positively in* C, *i.e.* $x_j \in C$
- ϕ *if* x_j *appears negatively in* C, *i.e.* $\overline{x}_j \in C$

Examples

Example 5 *If* $C = x_1\overline{x}_4x_6$,

- $C_{x_2} = C$ *(x_2 does not appear in C)*
- $C_{x_1} = \overline{x}_4x_6$ *(x_1 appears positively in C)*
- $C_{x_4} = \phi$ *(x_4 appears negatively in C)*

Example 6

$$\begin{aligned} F &= abc + \overline{b}d + \overline{c}d \\ F_b &= ac + \overline{c}d \end{aligned}$$

(*Just drop b everywhere and throw away cubes containing literal $\overline{b}$*).

Shannon Expansion

$f : B^n \rightarrow B$

Theorem 2 $f = x_i f_{x_i} + \overline{x}_i f_{\overline{x}_i}$

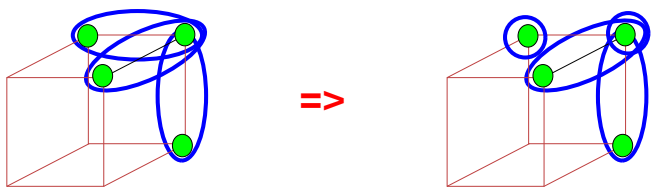
Theorem 3 *F is a cover of f. Then*

$$\widetilde{F} \equiv x_i F_{x_i} + \overline{x}_i F_{\overline{x}_i}$$

is a cover of f.

We say that f (F) is **expanded about** x_i. x_i is called the **splitting** variable.

Example 7 $F = ab + ac + bc$
$\widetilde{F} = aF_a + \overline{a}F_{\overline{a}} = a(b + c + bc) + \overline{a}(bc)$
$= ab + ac + abc + \overline{a}bc$

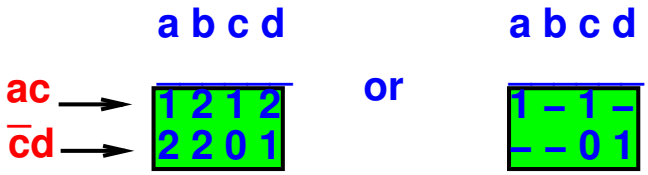


Cube bc got split into 2 cubes.

Cover Matrix

We sometimes use matrix notation to represent a cover:

Example 8 $F = ac + \bar{c}d =$



Each row represents a cube. 1 means that the positive literal appears in the cube, 0 the negative. The 2 here represents that the variable does **not appear** *in the cube. It also represents both 0 and 1 values.*

Example 9 $a\bar{c} = 1\ 2\ 0\ 2 = \{1, \{0,1\}, 0, \{0,1\}\}$
$= \{1000, 1100, 1001, 1101\}$

2 here is sometimes called "input don't care", but this is confusing so we won't use the term.

Incompletely Specified Functions

$\mathcal{F} = (f, d, r) : B^n \rightarrow \{0, 1, *\}$
where * represents "don't care". (*Sometimes we use 2 in place of **)

- $f =$ **onset** function - $f(x) = 1 \leftrightarrow \mathcal{F}(x) = 1$
- $r =$ **offset** function - $r(x) = 1 \leftrightarrow \mathcal{F}(x) = 0$
- $d =$ **don't care** function - $d(x) = 1 \leftrightarrow \mathcal{F}(x) = *$

(f, d, r) forms a **partition** of B^n, i.e.

1. $f^1 + d^1 + r^1 = B^n$
2. $f^1 d^1 = f^1 r^1 = d^1 r^1 = \phi$ - (*pairwise disjoint*)

A completely specified function g is a **cover** for $\mathcal{F} = (f, d, r)$ if

$$f \subseteq g \subseteq f + d$$

(*Thus* $gr = \phi$). Thus if $x \in d$ (i.e. $d(x) = 1$), then $g(x)$ can be 0 or 1, but if $x \in f$, then $g(x) = 1$ and if $x \in r$, then $g(x) = 0$.

(*We "don't care" which value* g *has at* $x \in d$)

Primes of Incompletely Specified Functions

Definition 5 *A cube c is* **prime** *of $\mathcal{F} = (f, d, r)$ if $c \subseteq f + d$ (*an implicant of $f + d$*), and no other implicant (of $f + d$) contains c, i.e.*

$$\forall \widetilde{c},\ \widetilde{c} \subseteq f + d,\ c \not\subseteq \widetilde{c}$$

*(*a prime of $f + d$*)*

Definition 6 *Cube c_j of cover $F = \{c_i\}$ is* **redundant** *if*

$$f \subseteq F - \{c_j\}$$

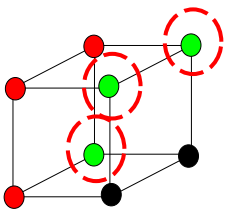
Otherwise it is **irredundant**.

Note that $c \subseteq f + d \leftrightarrow cr = \phi$.

Example - Logic Minimization

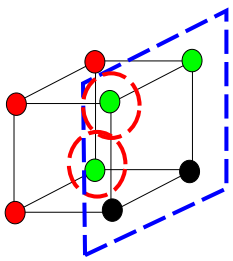
Consider $\mathcal{F}(a,b,c) = (f,d,r)$, where $f = \{\overline{a}b\overline{c}, a\overline{b}c, abc\}$ and $d = \{a\overline{b}\overline{c}, ab\overline{c}\}$, and the sequence of covers illustrated below:

- off
- on
- don't care



$F^1 = abc + a\overline{b}c + \overline{a}\overline{b}\overline{c}$

EXPAND $abc \rightarrow a$

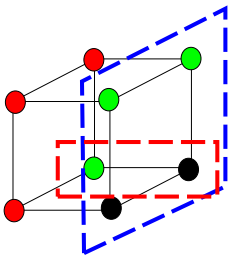


$F^2 = a + a\overline{b}c + \overline{a}\overline{b}\overline{c}$

$a\overline{b}c$ is redundant
a is prime

$F^3 = a + \overline{a}\overline{b}\overline{c}$

EXPAND $\overline{a}\overline{b}\overline{c} \rightarrow b\overline{c}$



$F^4 = a + b\overline{c}$

Checking for Prime and Irredundant

Let $G = \{c_i\}$ be a cover of $\mathcal{F} = (f, d, r)$. Let D be a cover for d.

1. $c_i \subseteq G$ is **redundant** iff

$$c_i \subseteq (G - \{c_i\}) \cup D \equiv G^i \qquad (1)$$

(*Since* $c_i \subseteq G^i$, *and* $f \subseteq G \subseteq f + d$ *then* $c_i = c_i f + c_i d$ *and* $c_i f \subseteq G - \{c_i\}$. *Thus* $f \subseteq G - \{c_i\}$)

2. A literal $l \in c_i$ is **prime** if $(c_i - \{l\}) = (c_i)_l$ is not an implicant of $\mathcal{F}$.
A cube c_i is a prime of $\mathcal{F}$ iff all literals $l \in c_i$ are prime.

$$\texttt{literal } l \in c_i \texttt{ is not prime} \iff (c_i)_l \subseteq f + d \quad (2)$$

Note: Both tests (1) and (2) can be checked by **tautology**:

1. $(G^i)_{c_i} \equiv 1$ (*implies* c_i *redundant*)

2. $(F \cup D)_{(c_i)_l} \equiv 1$ (*implies* l *not prime*)

Example

$$\begin{aligned} F &= (\overline{a}b\overline{c}, a\overline{b}c, abc) \\ D &= (a\overline{c}) \\ R &= (\overline{a}\overline{b}, \overline{a}c) \end{aligned}$$

Expand $abc \rightarrow ab = (c_i)_l = (abc)_c$
Check $ab \subseteq f + d \Longleftrightarrow (f + d)_{ab} = 1$
- $(f + d)_{ab} = c + \overline{c} \equiv 1$ **OK**

Expand $ab \rightarrow a$
Check $a \subseteq f + d$
- $(f + d)_a = \overline{b}c + bc + \overline{c} \equiv 1$ **OK**

$F = a + a\overline{b}c + \overline{a}b\overline{c}$

Check if $a\overline{b}c$ is redundant
- $a\overline{b}c \subseteq a + \overline{a}b\overline{c} + a\overline{c} = F - \{a\overline{b}c\} \cup D$
- $(a + \overline{a}b\overline{c} + a\overline{c})_{a\overline{b}c} = 1$ **OK**

Cover is now $F = a + \overline{a}b\overline{c}$ and $D = a\overline{c}$

Check if $\overline{a}b\overline{c}$ is redundant
$\overline{a}b\overline{c} \subseteq a + a\overline{c} = F - \{\overline{a}b\overline{c}\} \cup D$
- $(a + a\overline{c})_{\overline{a}b\overline{c}} = \phi$ **NOT REDUNDANT**